A Type- and Control-Flow Analysis for System F

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Introduction

Control-flow analysis

A compile-time approximation of the "flow" of functions in program: which functions might be bound to a given variable at run time.

- an enabling analysis for the compilation of functional languages
  - because control flow is not syntactically apparent
- typically formulated for dynamically- or simply-typed languages

- popular statically-typed functional languages are richly-typed
- System F (and extensions) are popular intermediate languages

Seek a control-flow analysis formulated for System F.

Exploit well-typedness to obtain more precise control-flow information.

Example 1

$f1 = \lambda x1.\ \cdots$
$f2 = \lambda x2.\ \cdots$

$id = \lambda x.\ x$

$res1 = id\ f1$
$res2 = id\ f2$

Example 1

$f1 = \lambda x1 . \ \cdots$
$f2 = \lambda x2 . \ \cdots$

$id = \lambda x . \ x$

$res1 = id\ f1$
$res2 = id\ f2$

	0CFA
$\hat{\rho}(x)$	$\{ f1 , f2 \}$
$\hat{\rho}(res1)$	$\{ f1 , f2 \}$
$\hat{\rho}(res2)$	$\{ f1 , f2 \}$

Example 1

$\mathsf{f1} : \mathsf{Int} \to \mathsf{Int} = \lambda \mathsf{x1} : \mathsf{Int} .\ \cdots$
$\mathsf{f2} : \mathsf{Bool} \to \mathsf{Bool} = \lambda \mathsf{x2} : \mathsf{Bool} .\ \cdots$

$\mathsf{id} : \forall \alpha .\ \alpha \to \alpha = \Lambda \alpha .\ \lambda \mathsf{x} : \alpha .\ \mathsf{x}$

$\mathsf{res1} : \mathsf{Int} \to \mathsf{Int} = \mathsf{id}\ [\mathsf{Int} \to \mathsf{Int}]\ \mathsf{f1}$
$\mathsf{res2} : \mathsf{Bool} \to \mathsf{Bool} = \mathsf{id}\ [\mathsf{Bool} \to \mathsf{Bool}]\ \mathsf{f2}$

	0CFA
$\hat{\rho}(\mathsf{x}^{\alpha})$	$\{ \mathsf{f1}^{\mathsf{Int}\to\mathsf{Int}}, \mathsf{f2}^{\mathsf{Bool}\to\mathsf{Bool}} \}$
$\hat{\rho}(\mathsf{res1}^{\mathsf{Int}\to\mathsf{Int}})$	$\{ \mathsf{f1}^{\mathsf{Int}\to\mathsf{Int}}, \mathsf{f2}^{\mathsf{Bool}\to\mathsf{Bool}} \}$
$\hat{\rho}(\mathsf{res2}^{\mathsf{Bool}\to\mathsf{Bool}})$	$\{ \mathsf{f1}^{\mathsf{Int}\to\mathsf{Int}}, \mathsf{f2}^{\mathsf{Bool}\to\mathsf{Bool}} \}$

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$\mathsf{res1} : \mathsf{Int} \to \mathsf{Int} = \mathsf{id}\ [\mathsf{Int} \to \mathsf{Int}]\ \mathsf{f1}$
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	0CFA then type-filter
$\hat{\rho}(\mathsf{x}^{\alpha})$	$\{ \mathsf{f1}^{\mathsf{Int}\to\mathsf{Int}}, \mathsf{f2}^{\mathsf{Bool}\to\mathsf{Bool}} \}$
$\hat{\rho}(\mathsf{res1}^{\mathsf{Int}\to\mathsf{Int}})$	$\{ \mathsf{f1}^{\mathsf{Int}\to\mathsf{Int}} \}$
$\hat{\rho}(\mathsf{res2}^{\mathsf{Bool}\to\mathsf{Bool}})$	$\{ \mathsf{f2}^{\mathsf{Bool}\to\mathsf{Bool}} \}$

Example 2

$$\mathsf{f1}:\mathsf{Int}\to\mathsf{Int}\to\mathsf{Int} = \lambda \mathsf{x1}:\mathsf{Int}.\ \texttt{let}\ \mathsf{g1}:\mathsf{Int}\to\mathsf{Int} = \lambda \mathsf{y1}:\mathsf{Int}.\ \cdots\ \texttt{in}\ \mathsf{g1}$$
$$\mathsf{f2}:\mathsf{Int}\to\mathsf{Int}\to\mathsf{Int} = \lambda \mathsf{x2}:\mathsf{Int}.\ \texttt{let}\ \mathsf{g2}:\mathsf{Int}\to\mathsf{Int} = \lambda \mathsf{y2}:\mathsf{Int}.\ \cdots\ \texttt{in}\ \mathsf{g2}$$
$$\mathsf{f3}:\mathsf{Int}\to\mathsf{Int}\to\mathsf{Int} = \lambda \mathsf{x3}:\mathsf{Int}.\ \texttt{let}\ \mathsf{g3}:\mathsf{Int}\to\mathsf{Int} = \lambda \mathsf{y3}:\mathsf{Int}.\ \cdots\ \texttt{in}\ \mathsf{g3}$$

$$\mathsf{id}:\forall\alpha.\ \alpha\to\alpha = \Lambda\alpha.\ \lambda \mathsf{x}:\alpha.\ \mathsf{x}$$

$$\begin{aligned}
&\mathsf{h}:\forall\beta.\ (\beta\to\mathsf{Int}\to\mathsf{Int})\to\beta\to\mathsf{Int}\to\mathsf{Int} = \\
&\quad \Lambda\beta.\ \lambda \mathsf{f}:\beta\to\mathsf{Int}\to\mathsf{Int}.\ \lambda \mathsf{y}:\beta. \\
&\quad \texttt{let}\ \mathsf{ff}:\beta\to\mathsf{Int}\to\mathsf{Int} = \mathsf{id}\ [\beta\to\mathsf{Int}\to\mathsf{Int}]\ \mathsf{f} \\
&\qquad\ \ \mathsf{g}:\mathsf{Int}\to\mathsf{Int} = \mathsf{ff}\ \mathsf{y} \\
&\quad \texttt{in}\ \mathsf{g} \\
&\cdots
\end{aligned}$$

Example 2

$$\mathsf{f1}:\mathsf{Int}\to\mathsf{Int}\to\mathsf{Int} = \lambda \mathsf{x1}:\mathsf{Int}.\ \texttt{let}\ \mathsf{g1}:\mathsf{Int}\to\mathsf{Int} = \lambda \mathsf{y1}:\mathsf{Int}.\ \cdots\ \texttt{in}\ \mathsf{g1}$$
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	0CFA
$\hat{\rho}(\mathsf{x}^{\alpha})$	$\{\ \mathsf{f1}^{\mathsf{Int}\to\mathsf{Int}\to\mathsf{Int}},\ \mathsf{f2}^{\mathsf{Bool}\to\mathsf{Int}\to\mathsf{Int}},\ \mathsf{f3}^{\mathsf{Str}\to\mathsf{Int}\to\mathsf{Int}}\ \}$
$\hat{\rho}(\mathsf{f}^{\beta\to\mathsf{Int}\to\mathsf{Int}})$	$\{\ \mathsf{f1}^{\mathsf{Int}\to\mathsf{Int}\to\mathsf{Int}},\ \mathsf{f2}^{\mathsf{Bool}\to\mathsf{Int}\to\mathsf{Int}}\ \}$
$\hat{\rho}(\mathsf{ff}^{\beta\to\mathsf{Int}\to\mathsf{Int}})$	$\{\ \mathsf{f1}^{\mathsf{Int}\to\mathsf{Int}\to\mathsf{Int}},\ \mathsf{f2}^{\mathsf{Bool}\to\mathsf{Int}\to\mathsf{Int}},\ \mathsf{f3}^{\mathsf{Str}\to\mathsf{Int}\to\mathsf{Int}}\ \}$
$\hat{\rho}(\mathsf{g}^{\mathsf{Int}\to\mathsf{Int}})$	$\{\ \mathsf{g1}^{\mathsf{Int}\to\mathsf{Int}},\ \mathsf{g2}^{\mathsf{Int}\to\mathsf{Int}},\ \mathsf{g3}^{\mathsf{Int}\to\mathsf{Int}}\ \}$

Example 2

$\mathsf{f1}:\mathsf{Int}\to\mathsf{Int}\to\mathsf{Int} = \lambda \mathsf{x1}:\mathsf{Int}.\ \texttt{let}\ \mathsf{g1}:\mathsf{Int}\to\mathsf{Int} = \lambda \mathsf{y1}:\mathsf{Int}.\ \cdots\ \texttt{in}\ \mathsf{g1}$
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$\qquad\mathsf{g}:\mathsf{Int}\to\mathsf{Int} = \mathsf{ff}\ \mathsf{y}$
$\quad\texttt{in}\ \mathsf{g}$
$\ldots$

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$\hat{\rho}(\mathsf{x}^{\alpha})$	$\{\mathsf{f1}^{\mathsf{Int}\to\mathsf{Int}\to\mathsf{Int}}, \mathsf{f2}^{\mathsf{Bool}\to\mathsf{Int}\to\mathsf{Int}}, \mathsf{f3}^{\mathsf{Str}\to\mathsf{Int}\to\mathsf{Int}}\}$
$\hat{\rho}(\mathsf{f}^{\beta\to\mathsf{Int}\to\mathsf{Int}})$	$\{\mathsf{f1}^{\mathsf{Int}\to\mathsf{Int}\to\mathsf{Int}}, \mathsf{f2}^{\mathsf{Bool}\to\mathsf{Int}\to\mathsf{Int}}\}$
$\hat{\rho}(\mathsf{ff}^{\beta\to\mathsf{Int}\to\mathsf{Int}})$	$\{\mathsf{f1}^{\mathsf{Int}\to\mathsf{Int}\to\mathsf{Int}}, \mathsf{f2}^{\mathsf{Bool}\to\mathsf{Int}\to\mathsf{Int}}, \mathsf{f3}^{\mathsf{Str}\to\mathsf{Int}\to\mathsf{Int}}\}$
$\hat{\rho}(\mathsf{g}^{\mathsf{Int}\to\mathsf{Int}})$	$\{\mathsf{g1}^{\mathsf{Int}\to\mathsf{Int}}, \mathsf{g2}^{\mathsf{Int}\to\mathsf{Int}}, \mathsf{g3}^{\mathsf{Int}\to\mathsf{Int}}\}$
$\hat{\phi}(\alpha)$	$\{\beta\to\mathsf{Int}\to\mathsf{Int}, \mathsf{Str}\to\mathsf{Int}\to\mathsf{Int}\}$
$\hat{\phi}(\beta)$	$\{\mathsf{Int}, \mathsf{Bool}\}$

Example 2

$$\mathsf{f1}:\mathsf{Int}\to\mathsf{Int}\to\mathsf{Int} = \lambda \mathsf{x1}:\mathsf{Int}.\ \texttt{let}\ \mathsf{g1}:\mathsf{Int}\to\mathsf{Int} = \lambda \mathsf{y1}:\mathsf{Int}.\ \cdots\ \texttt{in}\ \mathsf{g1}$$
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$$\texttt{let}\ \mathsf{ff}:\beta\to\mathsf{Int}\to\mathsf{Int} = \mathsf{id}\ [\beta\to\mathsf{Int}\to\mathsf{Int}]\ \mathsf{f}$$
$$\mathsf{g}:\mathsf{Int}\to\mathsf{Int} = \mathsf{ff}\ \mathsf{y}$$
$$\texttt{in}\ \mathsf{g}$$
$$\ldots$$

	0T&CFA
$\hat{\rho}(\mathsf{x}^{\alpha})$	$\{\mathsf{f1}^{\mathsf{Int}\to\mathsf{Int}\to\mathsf{Int}}, \mathsf{f2}^{\mathsf{Bool}\to\mathsf{Int}\to\mathsf{Int}}, \mathsf{f3}^{\mathsf{Str}\to\mathsf{Int}\to\mathsf{Int}}\}$
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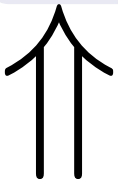
Introduction

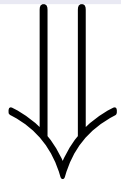
A type- and control-flow analysis for System F

Type-flow analysis

A compile-time approximation of the "flow" of types in program: which types might be bound to a given type variable at run time



determines type abstractions flowing to type applications



rejects flows incompatible with static typing

Control-flow analysis

A compile-time approximation of the "flow" of functions in program: which functions might be bound to a given variable at run time

System F (in Administrative Normal Form (ANF))

$$\mathit{Type} \ni \tau ::= \tau \to \tau \mid \alpha \mid \forall\alpha.\, \tau$$

$$\begin{aligned}
\mathit{Exp} \ni e ::= {} & x \\
\mid {} & \texttt{let } x{:}\tau_x = \mu f{:}\tau_f.\lambda z{:}\tau_z.e_b \texttt{ in } e \\
\mid {} & \texttt{let } x{:}\tau_x = \mu f{:}\tau_f.\Lambda\beta.e_b \texttt{ in } e \\
\mid {} & \texttt{let } x{:}\tau_x = x_f\ x_a \texttt{ in } e \\
\mid {} & \texttt{let } x{:}\tau_x = x_f\ [\tau_a] \texttt{ in } e
\end{aligned}$$

- Type system — standard
- Operational semantics — C_aEK abstract machine

Type- and Control-Flow Analysis

$$\begin{array}{lcll} \mathit{AType} & \ni & \hat{\pi} & ::= \quad \tau \\ \mathit{ATEnv} & \ni & \hat{\phi} & \in \quad \mathit{TyVar} \to \mathcal{P}(\mathit{AType}) \end{array} \qquad \begin{array}{lcll} \mathit{AValue} & \ni & \hat{w} & ::= \quad \mu f{:}\tau_f\,.\,\lambda z{:}\tau_z\,.\,e_b \mid \\ & & & \quad\ \mu f{:}\tau_f\,.\,\Lambda\beta\,.\,e_b \\ \mathit{AEnv} & \ni & \hat{\rho} & \in \quad \mathit{Var} \to \mathcal{P}(\mathit{AValue}) \end{array}$$

$\boxed{\hat{\phi}; \hat{\rho} \vDash e}$ "$\hat{\phi}$ and $\hat{\rho}$ are an *acceptable* type- and control-flow analysis for e"

Type- and Control-Flow Analysis

$$\begin{array}{llllll} \mathit{AType} & \ni & \hat{\pi} & ::= & \tau & \\ \mathit{ATEnv} & \ni & \hat{\phi} & \in & \mathit{TyVar} \to \mathcal{P}(\mathit{AType}) & \end{array} \qquad \begin{array}{llllll} \mathit{AValue} & \ni & \hat{w} & ::= & \mu f:\tau_f\,.\,\lambda z:\tau_z\,.\,e_b \mid \\ & & & & \mu f:\tau_f\,.\,\Lambda\beta\,.\,e_b \\ \mathit{AEnv} & \ni & \hat{\rho} & \in & \mathit{Var} \to \mathcal{P}(\mathit{AValue}) \end{array}$$

$\boxed{\hat{\phi};\hat{\rho} \vDash e}$ "$\hat{\phi}$ and $\hat{\rho}$ are an *acceptable* type- and control-flow analysis for e"

$$\frac{}{\hat{\phi};\hat{\rho} \vDash x}$$

$$\frac{\hat{\rho}(f) \supseteq \{\mu f:\tau_f\,.\,\lambda z:\tau_z\,.\,e_b\} \qquad \hat{\phi};\hat{\rho} \vDash e_b \\ \hat{\rho}(x) \supseteq \{\mu f:\tau_f\,.\,\lambda z:\tau_z\,.\,e_b\} \qquad \hat{\phi};\hat{\rho} \vDash e}{\hat{\phi};\hat{\rho} \vDash \texttt{let}\ x:\tau_x = \mu f:\tau_f\,.\,\lambda z:\tau_z\,.\,e_b\ \texttt{in}\ e}$$

$$\frac{\hat{\rho}(f) \supseteq \{\mu f:\tau_f\,.\,\Lambda\beta\,.\,e_b\} \qquad \hat{\phi};\hat{\rho} \vDash e_b \\ \hat{\rho}(x) \supseteq \{\mu f:\tau_f\,.\,\Lambda\beta\,.\,e_b\} \qquad \hat{\phi};\hat{\rho} \vDash e}{\hat{\phi};\hat{\rho} \vDash \texttt{let}\ x:\tau_x = \mu f:\tau_f\,.\,\Lambda\beta\,.\,e_b\ \texttt{in}\ e}$$

Type- and Control-Flow Analysis

$$\mathit{AType} \ni \hat{\pi} ::= \tau \qquad \mathit{AValue} \ni \hat{w} ::= \mu f:\tau_f\,.\,\lambda z:\tau_z\,.\,e_b \mid \mu f:\tau_f\,.\,\Lambda\beta\,.\,e_b$$

$$\mathit{ATEnv} \ni \hat{\phi} \in \mathit{TyVar} \to \mathcal{P}(\mathit{AType}) \qquad \mathit{AEnv} \ni \hat{\rho} \in \mathit{Var} \to \mathcal{P}(\mathit{AValue})$$

$\boxed{\hat{\phi};\hat{\rho} \vDash e}$ "$\hat{\phi}$ and $\hat{\rho}$ are an *acceptable* type- and control-flow analysis for e"

$$\frac{\bigwedge_{\mu f:\tau_f\,.\,\lambda z:\tau_z\,.\,e_b \in \hat{\rho}(x_f)} \begin{pmatrix} \hat{\rho}(z) \supseteq \hat{\rho}(x_a) \wedge \\ \hat{\rho}(x) \supseteq \hat{\rho}(\mathsf{last}(e_b)) \end{pmatrix} \qquad \hat{\phi};\hat{\rho} \vDash e}{\hat{\phi};\hat{\rho} \vDash \texttt{let } x:\tau_x = x_f\ x_a \texttt{ in } e}$$

$$\frac{\bigwedge_{\mu f:\tau_f\,.\,\Lambda\beta\,.\,e_b \in \hat{\rho}(x_f)} \begin{pmatrix} \hat{\phi}(\beta) \supseteq \{\tau_a\} \wedge \\ \hat{\rho}(x) \supseteq \hat{\rho}(\mathsf{last}(e_b)) \end{pmatrix} \qquad \hat{\phi};\hat{\rho} \vDash e}{\hat{\phi};\hat{\rho} \vDash \texttt{let } x:\tau_x = x_f\ [\tau_a] \texttt{ in } e}$$

Type- and Control-Flow Analysis

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$\boxed{\hat{\phi};\hat{\rho} \vDash e}$ "$\hat{\phi}$ and $\hat{\rho}$ are an *acceptable* type- and control-flow analysis for e"

$$\frac{\bigwedge_{\mu f:\tau_f.\lambda z:\tau_z.e_b \in \hat{\rho}(x_f)} \begin{pmatrix} \hat{\rho}(z) \supseteq \{\hat{w}_a \in \hat{\rho}(x_a) \mid \vdash \hat{\phi} \Rrightarrow \hat{w}_a :\approx \tau_z\} \wedge \\ \hat{\rho}(x) \supseteq \{\hat{w}_b \in \hat{\rho}(\mathsf{last}(e_b)) \mid \vdash \hat{\phi} \Rrightarrow \hat{w}_b :\approx \tau_x\} \end{pmatrix} \qquad \hat{\phi};\hat{\rho} \vDash e}{\hat{\phi};\hat{\rho} \vDash \texttt{let } x:\tau_x = x_f\ x_a \texttt{ in } e}$$

$$\frac{\bigwedge_{\mu f:\tau_f.\Lambda\beta.e_b \in \hat{\rho}(x_f)} \begin{pmatrix} \hat{\phi}(\beta) \supseteq \{\tau_a\} \wedge \\ \hat{\rho}(x) \supseteq \{\hat{w}_b \in \hat{\rho}(\mathsf{last}(e_b)) \mid \vdash \hat{\phi} \Rrightarrow \hat{w}_b :\approx \tau_x\} \end{pmatrix} \qquad \hat{\phi};\hat{\rho} \vDash e}{\hat{\phi};\hat{\rho} \vDash \texttt{let } x:\tau_x = x_f\ [\tau_a] \texttt{ in } e}$$

Type Compatibility

$\boxed{\vdash \hat{\phi} \twoheadrightarrow \hat{w} :\approx \hat{\pi}}$ "the type of $\hat{w}$ is *compatible* with $\hat{\pi}$ under $\hat{\phi}$"

$$\frac{\emptyset \vdash \hat{\phi} \twoheadrightarrow \tau_f \approx \hat{\pi}}{\vdash \hat{\phi} \twoheadrightarrow \mu f : \tau_f \,.\, \lambda z : \tau_z \,.\, e_b :\approx \hat{\pi}} \qquad \frac{\emptyset \vdash \hat{\phi} \twoheadrightarrow \tau_f \approx \hat{\pi}}{\vdash \hat{\phi} \twoheadrightarrow \mu f : \tau_f \,.\, \Lambda \beta \,.\, e_b :\approx \hat{\pi}}$$

$\boxed{\Delta \vdash \hat{\phi} \twoheadrightarrow \hat{\pi}_1 \approx \hat{\pi}_2}$ "$\hat{\pi}_1$ and $\hat{\pi}_2$ are *compatible* under $\hat{\phi}$"

$$\frac{\hat{\pi}_1' \in \hat{\phi}(\alpha_1) \qquad \mathsf{FTV}(\hat{\pi}_2) \cap \mathsf{dom}(\Delta) = \emptyset \qquad \emptyset \vdash \hat{\phi} \twoheadrightarrow \hat{\pi}_1' \approx \hat{\pi}_2}{\Delta \vdash \hat{\phi} \twoheadrightarrow \alpha_1 \approx \hat{\pi}_2} \qquad \frac{\hat{\pi}_2' \in \hat{\phi}(\alpha_2) \qquad \mathsf{FTV}(\hat{\pi}_1) \cap \mathsf{dom}(\Delta) = \emptyset \qquad \emptyset \vdash \hat{\phi} \twoheadrightarrow \hat{\pi}_1 \approx \hat{\pi}_2'}{\Delta \vdash \hat{\phi} \twoheadrightarrow \hat{\pi}_1 \approx \alpha_2}$$

$$\frac{\Delta \vdash \hat{\phi} \twoheadrightarrow \tau_{z1} \approx \tau_{z2} \qquad \Delta \vdash \hat{\phi} \twoheadrightarrow \tau_{b1} \approx \tau_{b2}}{\Delta \vdash \hat{\phi} \twoheadrightarrow \tau_{z1} \to \tau_{b1} \approx \tau_{z2} \to \tau_{b2}}$$

$$\frac{\Delta(\alpha) = \star}{\Delta \vdash \hat{\phi} \twoheadrightarrow \alpha \approx \alpha} \qquad \frac{\Delta, \alpha{:}\star \vdash \hat{\phi} \twoheadrightarrow \tau_{b1} \approx \tau_{b2}}{\Delta \vdash \hat{\phi} \twoheadrightarrow \forall \alpha.\ \tau_{b1} \approx \forall \alpha.\ \tau_{b2}}$$

Flow Soundness

Acceptable abstract environments approximate concrete environments

(for well-typed programs)

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Theorem (Flow Soundness)

If $\emptyset;\emptyset \vdash e : \tau$, $\hat{\phi};\hat{\rho} \vDash e$, *and* $\langle e;\emptyset;\emptyset;\bullet\rangle \longrightarrow^* \langle e';\phi';\rho';\kappa'\rangle$,
then $|\phi'| \sqsubseteq \hat{\phi}$ *and* $|\rho'| \sqsubseteq \hat{\rho}$.

- syntactic preservation proof (with $\vdash \langle e;\phi;\rho;\kappa\rangle : \tau$ and $\hat{\phi};\hat{\rho} \vDash \langle e;\phi;\rho;\kappa\rangle$)

Lemma (Expansion Type Equality implies Analysis-Time Type Compatibility)

If $\vdash \langle \tau_1;\phi_1\rangle \Rrightarrow \tau$, $\vdash \langle \tau_2;\phi_2\rangle \Rrightarrow \tau$, $\hat{\phi} \vDash \phi_1$, *and* $\hat{\phi} \vDash \phi_2$, *then* $\emptyset \vdash \hat{\phi} \Rrightarrow \tau_1 \approx\!\!\!\!\approx \tau_2$.

- $\vdash \langle \tau_1;\phi_1\rangle \Rrightarrow \tau$ and $\vdash \langle \tau_2;\phi_2\rangle \Rrightarrow \tau$ from well-typedness

Existence of Minimum, Finite Flows

There are “best” acceptable abstract environments

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There are "best" acceptable abstract environments

Theorem (Minimum Flows Exist)

For all programs e,
there exist minimum abstract environments $\hat{\phi}_{\mathsf{min}}$ *and* $\hat{\rho}_{\mathsf{min}}$
such that $\hat{\phi}_{\mathsf{min}}; \hat{\rho}_{\mathsf{min}} \vDash e$.

- acceptable abstract environments are preserved by meets

Existence of Minimum, Finite Flows

There are "best" acceptable abstract environments

Theorem (Finite Flows Exist)

For all programs e, $\hat{\phi}^e_\top; \hat{\rho}^e_\top \vDash e$.

- $\hat{\phi}^e_\top$ maps every Λ-bound type variable in e (a finite set) to the (finite) set of all type expressions in e
- $\hat{\rho}^e_\top$ maps every `let`-, μ-, and λ-bound variable in e (a finite set) to the (finite) set of all Λ- and λ-expressions in e

Computability/Decidability of Flows

Are the "best" acceptable abstract environments computable?

Computability/Decidability of Flows

Are the "best" acceptable abstract environments computable?

Is the acceptability of abstract environments decidable?

- To show $\hat{\phi}; \hat{\rho} \vDash e$,
  must show inclusions of the form $\hat{\rho}(z) \supseteq \{\hat{w} \in \hat{\rho}(x) \mid \vdash \hat{\phi} \twoheadrightarrow \hat{w} :\approx \tau_z\}$
  - For each $\hat{w} \in \hat{\rho}(x)$ but $\hat{w} \notin \hat{\rho}(z)$,
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Is type compatibility decidable?

Decidability of Type Compatibility

If $\hat{\phi}^{\ddagger}(\alpha) = \{\, \mathsf{int} \to \mathsf{int}\,,\ \mathsf{int} \to \alpha \,\}$ and $\hat{\phi}^{\ddagger}(\beta) = \{\, \mathsf{int} \to \mathsf{bool}\,,\ \mathsf{int} \to \beta \,\}$,
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$$\dfrac{\mathsf{int} \to \alpha \in \hat{\phi}^{\ddagger}(\alpha) \qquad \dfrac{\dfrac{}{\emptyset \vdash \hat{\phi}^{\ddagger} \twoheadrightarrow \mathsf{int} \approx\!\!\!\!\approx \mathsf{int}} \qquad \dfrac{\mathsf{int} \to \beta \in \hat{\phi}^{\ddagger}(\beta) \qquad \dfrac{\vdots}{\emptyset \vdash \hat{\phi}^{\ddagger} \twoheadrightarrow \mathsf{int} \to \beta \approx\!\!\!\!\approx \alpha}}{\emptyset \vdash \hat{\phi}^{\ddagger} \twoheadrightarrow \beta \approx\!\!\!\!\approx \alpha}}{\emptyset \vdash \hat{\phi}^{\ddagger} \twoheadrightarrow \mathsf{int} \to \beta \approx\!\!\!\!\approx \mathsf{int} \to \alpha}}{\emptyset \vdash \hat{\phi}^{\ddagger} \twoheadrightarrow \mathsf{int} \to \beta \approx\!\!\!\!\approx \alpha}$$

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Interpret abstract type environment as a regular-tree grammar.

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Interpret abstract type environment as a regular-tree grammar.

$$\begin{aligned} \mathcal{L}_{\hat{\phi}^{\ddagger}}(\mathsf{int} \to \beta) &= \{\mathsf{int} \to \mathsf{int} \to \mathsf{bool}, \mathsf{int} \to \mathsf{int} \to \mathsf{int} \to \mathsf{bool}, \ldots\} \\ \mathcal{L}_{\hat{\phi}^{\ddagger}}(\alpha) &= \{\mathsf{int} \to \mathsf{int}, \mathsf{int} \to \mathsf{int} \to \mathsf{int}, \mathsf{int} \to \mathsf{int} \to \mathsf{int} \to \mathsf{int}, \ldots\} \end{aligned}$$

$$\mathcal{L}_{\hat{\phi}^{\ddagger}}(\mathsf{int} \to \beta) \cap \mathcal{L}_{\hat{\phi}^{\ddagger}}(\alpha) = \emptyset$$

Intuitively, $\emptyset \vdash \hat{\phi}^{\ddagger} \Rrightarrow \mathsf{int} \to \beta \approx\!\!\!\!\approx \alpha$ is not derivable because there is no closed type generated by $\hat{\phi}^{\ddagger}$ from both $\mathsf{int} \to \beta$ and α.

Decidability of Type Compatibility

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Theorem (Analysis-Time Type Compatibility iff Languages Intersect)

$\emptyset \vdash \hat{\phi} \Rrightarrow \hat{\pi}_1 \approxeq \hat{\pi}_2$ *if and only if* $\mathcal{L}_{\hat{\phi}}(\hat{\pi}_1) \cap \mathcal{L}_{\hat{\phi}}(\hat{\pi}_2) \neq \emptyset$.

- regular-tree grammars are closed under intersection; $O(n^2)$
- emptiness of regular-tree grammars is decidable; $O(n^2)$

Computability of Flows

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The "best" acceptable abstract environments are computable

- Read analysis as a monotone function
  from abstract environments to abstract environments
- Least fixed point is the "best" acceptable abstract environments
- Least fixed point is computable using iteration

Related Work

- Need to distinguish between
  - flow analyses expressed as sophisticated type systems (many)
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- Type-Directed Flow Analysis for Typed Intermediate Languages; Jagannathan, Weeks, & Wright (SAS'97)
  - limited to predicative System F with recursion
  - polyvariant analysis
  - diverges on programs using polymorphic recursion

- Type-sensitive Control-Flow Analysis; Reppy (ML'06)
  - Suggests mapping polymorphism to $\top$

Future Work

- Efficient computation of type- and control-flow analysis
- Extend to a polyvariant type- and control-flow analysis
- Extend to richer type systems: System F + GADTs; System F_ω

A Type- and Control-Flow Analysis for System F

Questions?