The Pumping Lemma

The Burning Question…

- We've looked at a number of regular languages
- I know that you are just dying to know…
  - Is there a language L that is not regular?
  - To answer this, we'll use what is known as The Pumping Lemma.

The Pumping Lemma

- The pumping lemma formalizes the idea that if a string from a RL is long enough, eventually at least one state on its FA will be have to be repeated on the path that accepts the string.
  - Implies that there is a Kleene star in there somewhere!
- Continually looping on this state will produce an infinite number of strings in the language

The Pumping Lemma

- Statement of the pumping lemma
  - Let L be a regular language.
  - Then there exists a constant n (which varies for different languages), such that for every string $x \in L$ with $|x| \geq n$, x can be expressed as x = uvw such that:
    1. $|v| > 0$
    2. $|uv| \leq n$
    3. For all $k \geq 0$, the string uv^kw is also in L.

The Pumping Lemma

- What this means
  - For a long enough string x in L:
    - We can express x as the concatenation of three smaller strings
    - The middle string can be "pumped" (repeated) any number of times (including 0 = deleting) and the resulting string will be in L.

Pumping Lemma

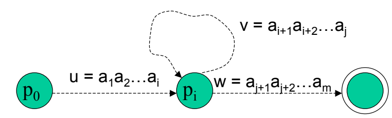
- Proof of the pumping lemma
  - Since L is regular, there is a FA $M=(Q,\Sigma,q_0,A,\delta)$ that accepts L.
    - Assume M has n states.
  - Consider a string x with $|x| = m \geq n$.
    - Express $x = a_1a_2a_3 \ldots a_m$ where each $a_i \in \Sigma$.
  - Define p_i to be the state M is in after reading i characters:
    - $p_i = \hat{\delta}(q_0, a_1a_2\ldots a_i)$
    - $p_0 = q_0$

Pumping Lemma

- Proof of the pumping lemma
  - Since $|x| \geq n$, and we only have n states, one state on it's path must visited more than once.
    - There exists integers i and j, $0 \leq i < j \leq n$ such that $p_i = p_j$
    - Then x = uvw
      - $u = a_1a_2\ldots a_i$
      - $v = a_{i+1}a_{i+2}\ldots a_j$
      - $w = a_{j+1}a_{j+2}\ldots a_m$

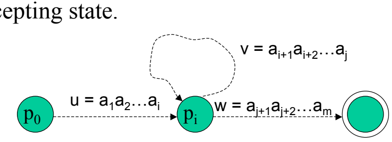
Pumping Lemma

- Proof of pumping lemma
  - Then x = uvw
    - $u = a_1a_2\ldots a_i$
    - $v = a_{i+1}a_{i+2}\ldots a_j$
    - $w = a_{j+1}a_{j+2}\ldots a_m$



Pumping Lemma

- Proof of pumping lemma
  - You can loop (pump) on the v loop 0 or more times and there will still be a path to the accepting state.



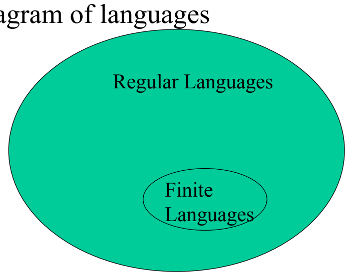
Pumping Lemma

- So what good is the pumping lemma?
- It can be used to answer that burning question:
  - Is there a language L that is <u>not</u> regular?

Non-regular languages

- Venn-diagram of languages

Is there something out here?



Pumping lemma

- The <u>real</u> strength of the pumping lemma is proving that languages are not regular
  - Proof by contradiction
    - Assume that the language to be tested is regular
    - Use the pumping lemma to come to a contradiction
    - Original assumption about the language being regular is false
- You <u>cannot</u> prove a language <u>to be</u> regular using the Pumping Lemma!!!!

Pumping lemma

- The Pumping Lemma game
  - To show that a language L is not regular
    - Assume L is regular
    - Choose an “appropriate” string x in L
      - In terms of n (number of states in DFA)
    - Express x = uvw following rules of pumping lemma
    - Show that uv^kw is not in L, for some k
    - The above contradicts the Pumping Lemma
    - Our assumption that L is regular is wrong
    - L must not be regular

Pumping Lemma

Pumping lemma

- Example:
  - $L = \{x \in \{0,1\}^* \mid 0^i1^i, i \geq 0\}$
  - Ex: 000111, 0011, ε, 00001111

Pumping lemma

- Example
  - Let’s play!
    - Assume that L is regular.
      - Then there is an FA, M that accepts L.
      - Let n be the number of states in M

Pumping lemma

- Example:
  - $L = \{x \in \{0,1\}^* \mid 0^i1^i, i \geq 0\}$
  - Let’s play
    - Choose an appropriate string $x \in L$
      - Let $x = 0^n1^n$
    - Apply Pumping Lemma to x
      - x = uvw
      - $|uv| \leq n$
      - $|v| > 0$

Pumping lemma

- $x = uvw = 0^n1^n$
  - 00 … 0 11 … 1
  - Since $|uv| \leq n$, uv must consists entirely of 0s and, as such, v must also consist entirely of 0s.
    - $v = 0^j$ for some $j \leq n$

Pumping lemma

- $x = uvw = 0^n1^n = 0^i0^j0^k1^n \quad i + j+k= n$
  - Let's pump!
  - By the Pumping Lemma
    - uv^2w is also in L
    - $uv^2w = 0^i0^{2j}0^k1^n$
    - Certainly $i + 2j + k \neq n$
    - uv^2w has more 0's that 1's
    - Thus $uv^2w \notin L$ CONTRADICTION!

Pumping lemma

- We arrived at a contradiction,
  - Thus our original assumption that L is regular must be incorrect
  - Thus L is not regular.
- Note that we need to find only 1 string x that fails in order for the proof by contradiction to work.
  - The key is finding the x that won't work
- Questions?

Pumping Lemma

- Let's try another example:
  - $L = \{x \in \{0,1\}^* \mid 0^ix, |x| \leq i\}$
  - Ex: 0001, 0010, ε, 0000101
  - Let's play!
    - Assume that L is regular.
      - Then there is an FA, M that accepts L.
      - Let n be the number of states in M

Pumping Lemma

- Another Example:
  - $L = \{x \in \{0,1\}^* \mid 0^ix, |x| \leq i\}$
  - Let's play
    - Choose an appropriate string $x \in L$
      - Let $x = 0^n1^n$
    - Apply Pumping Lemma to x
      - $x = uvw$
      - $|uv| \leq n$
      - $|v| > 0$

Pumping Lemma

- $x = uvw = 0^n1^n$
  - 00 … 0 11 … 1
  - Since $|uv| \leq n$, uv must consists entirely of 0s and, as such, v must also consist entirely of 0s.
    - $v = 0^j$ for some $j \leq n$

Pumping Lemma

- $x = uvw = 0^n1^n = 0^i0^j0^k1^n \quad i + j+k= n$
  - Let's (un)pump!
  - By the Pumping Lemma
    - uv^0w is also in L
    - $uv^0w = uw = 0^i0^k1^n$
    - Certainly $n > i + k$
    - The length of the prefix of 0s is less than the suffix x
    - Thus $uv^0w \notin L$ CONTRADICTION!

Pumping Lemma

- We arrived at a contradiction,
  - Thus our original assumption that L is regular must be incorrect
  - Thus L is not regular.
- Note that we need to find only 1 string x that fails in order for the proof by contradiction to work.
  - We can show x not to work by pumping 0 times
- Questions?

Non-regular languages

- Informal notion of what regular languages can't express:
  - Counting and comparing
  - Any operation that implies the use of a stack
    - Pal
    - xx^r

Pumping Lemma

- Let's try another example:
  - L = set of palindromes over {a,b}
    - Strings that read the same forwards and backwards
    - "Madam I'm Adam"
  - Ex: aa, abba, abbbbba, ε
  - Let's play!
    - Assume that L is regular.
      - Then there is an FA, M that accepts L.
      - Let n be the number of states in M

Pumping Lemma

- Another Example:
  - L = set of palindromes over {a,b}
  - Let's play
    - Choose an appropriate string $x \in L$
      - Let $x = a^n b a^n$
    - Apply Pumping Lemma to x
      - $x = uvw$
      - $|uv| \leq n$
      - $|v| > 0$

Pumping Lemma

- $x = uvw = a^n b a^n$
  - aa … a b aa … a
  - Since $|uv| \leq n$, uv must consists entirely of a and, as such, v must also consist entirely of a.
    - $v = a^j$ for some $j \leq n$

Pumping Lemma

- $x = uvw = a^n b a^n = a^i a^j a^k b a^n \quad i + j + k = n$
  - Let's pump!
  - By the Pumping Lemma
    - uv^2w is also in L
    - uv^2w has more than n a's
    - Number of a's following b is still n
    - Thus uv^2w cannot be a palidrome
    - Thus $uv^2w \notin L$ CONTRADICTION!

Pumping Lemma

- We arrived at a contradiction,
  - Thus our original assumption that L is regular must be incorrect
  - Thus L is not regular.
- Note that we need to find only 1 string x that fails in order for the proof by contradiction to work.
- Questions?

Pumping Lemma

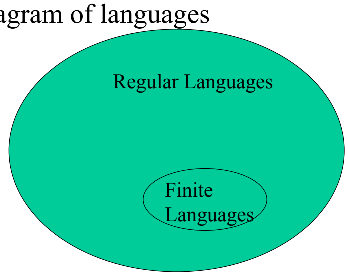
- Summary
  - The pumping lemma formalizes the idea that if a string is longer enough, eventually at least one state on the DFA will be have to be repeated on the path that accepts the string.
  - Continually looping on this state will produce an infinite number of strings in the language
  - Used to show that languages are not regular
  - Has other uses as we'll see next time.

Non-regular languages

- Venn-diagram of languages

Is there something out here?

YES



Next time

- Decision properties / algorithms for RLs
- Problem Session…
  - For homeworks: past and present
- Questions?