Accelerating Simulated Annealing for the Permanent and Combinatorial Counting Problems

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Accelerating Simulated Annealing for the Permanent and Combinatorial Counting Problems

Talk outline:

1. The Permanent problem
2. Simulated annealing for the Permanent (MCMC algorithm by JSV '01)
3. New simulated annealing schedule
**Permanent** of an $n \times n$ matrix $A$

\[
\text{Per}(A) = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)}
\]

**History & motivation:**

- defined by Cauchy [1812]
- used in a variety of areas: statistical physics, statistics, vision, anonymization systems, ...
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$A$ - binary (entries 0 or 1):
adjacency matrix of a bipartite graph
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adjacency matrix of a bipartite graph

The permanent counts the number of perfect matchings.
Previous Work on the Permanent Problem

[Kasteleyn '67]  
poly-time for planar graphs (bipartite or not)

[Valiant '79]  
#P-complete for non-planar graphs

[Jerrum-Sinclair '89]  
fpras for special graphs, e.g. the dense graphs, based on a Markov chain by Broder '88

[Jerrum-Sinclair-Vigoda '01 & '05]  
\(O^*(n^{26})\) fpras for any bipartite graph, later \(O^*(n^{10})\)

Our result:  
\(O^*(n^7)\)
Broder chain

uniform sampling of perfect matchings of a given graph

At a perfect matching:
  • remove a random edge

At a near-matching:
  • pick a vertex at random
    - if a hole, try to match with the other hole
    - otherwise slide (if can)
**Broder chain**

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Does the Broder chain mix in polynomial time?
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State space

Exponentially smaller!

Perfect matchings
Theorem [JS]: Rapid mixing if perfect matchings polynomially related to near-perfect matchings.
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**Idea [JSV]:** *Weight the states* so that the *weighted ratio* is always polynomially bounded.
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\( n^2 + 1 \) regions, very different weight
Theorem[JS]: Rapid mixing if perfect matchings polynomially related to near-perfect matchings.

Idea[JSV]: Weight the states so that the weighted ratio is always polynomially bounded.

\[ n^2 + 1 \text{ regions, each about the same weight} \]

Ideal weights
(for a matching with holes u,v):

\[ \frac{\text{(# perfects)}}{\text{(# nears with holes u,v)}} \]
Good: A perfect matching sampled with prob. $1/(n^2+1)$

Bad: Computing ideal weights as hard as original problem?

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Solution: Approximate the ideal weights

Ideal weights
(for a matching with holes $u,v$):

$\frac{(\# \text{ perfects})}{(\# \text{ nears with holes } u,v)}$
Simulated Annealing

Solution: *Approximate* the ideal weights

Start with an easy instance,
gradually get to the target instance.

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- start with the complete graphs (weights easy to compute)
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The edges have activities:

- 1 for a real edge
- \( \lambda \in [0,1] \) for a non-edge
How activities help?

• start with $\lambda = 1$

• compute corresponding weights $n! / (n-1)!$
How activities help?

- start with $\lambda = 1$
- compute corresponding weights $\frac{n!}{(n-1)!}$

Repeat until $\lambda < 1/n$!
Running Time [JSV]

Thm: The $(\lambda,\text{hole-weights})$-Broder chain mixes in time $O^*(n^6)$.

We need:

- $O^*(n^6)$ per sample
- $O^*(n^2)$ samples (boosting from 4-apx to 2-apx)
- $O^*(n^2)$ $\lambda$-decrements (phases)

$O^*(n^{10})$ total to get a 2-apx of the ideal weights
**Running Time [BŠVV]**

Thm: The \((\lambda,\text{hole-weights})\)-Broder chain mixes in time \(O^*(n^6)\).

We need:

<table>
<thead>
<tr>
<th>(O^*(n^4))</th>
<th>(O^*(n^6))</th>
<th>per sample</th>
</tr>
</thead>
<tbody>
<tr>
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</table>
Reformulation of the problem

Promise: a set of polynomials of degree $n$ such that

- polynomials have a low-degree term
- non-negative integer coefficients sum to $\leq n!$

Goal: $\lambda$-sequence (from 1 to $1/n!$) such that

for every polynomial ratio of consecutive values $\leq 2$

Tricky part:

Don't know the coefficients!
Intuition

The worst case is the set of polynomials $x^j, j=1,\ldots,n$

**Problem:** no low-degree terms and $x^n$ "dominates"

**Fix:** if the value of some $x^j$ drops below $1/n!$, ignore the polynomial

**TOTAL:** $O(n \log^2 n)$ points
Conclusions

- new cooling schedule: a blackbox, applicable to other problems
- improved analysis of the weighted Broder chain
- interest of practical community

Open Problems

- other applications of the cooling schedule
- faster mixing result
- do we need $n^2$ weights?
- non-bipartite graphs