Accelerating Simulated Annealing for the Permanent and Combinatorial Counting Problems

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Accelerating Simulated Annealing for the Permanent and Combinatorial Counting Problems

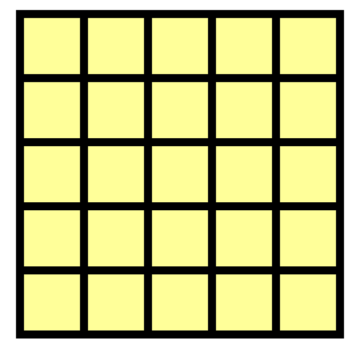
Talk outline:

1. The Permanent problem
2. Simulated annealing for the Permanent (MCMC algorithm by JSV '01)
3. New simulated annealing schedule

Permanent of an nxn matrix A

$$\mathrm{Per}(A) = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)}$$

A =



History & motivation:

- defined by Cauchy [1812]
- used in a variety of areas: statistical physics, statistics, vision, anonymization systems, …

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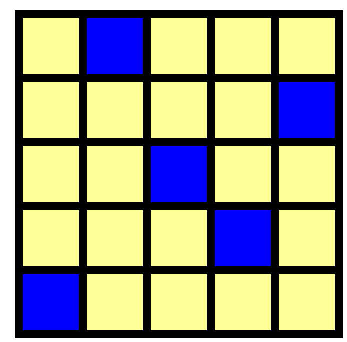
A = 

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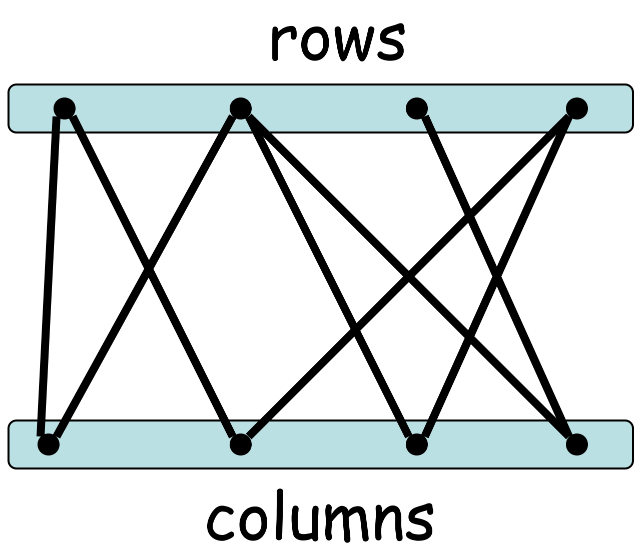
A - binary (entries 0 or 1):
adjacency matrix of a bipartite graph



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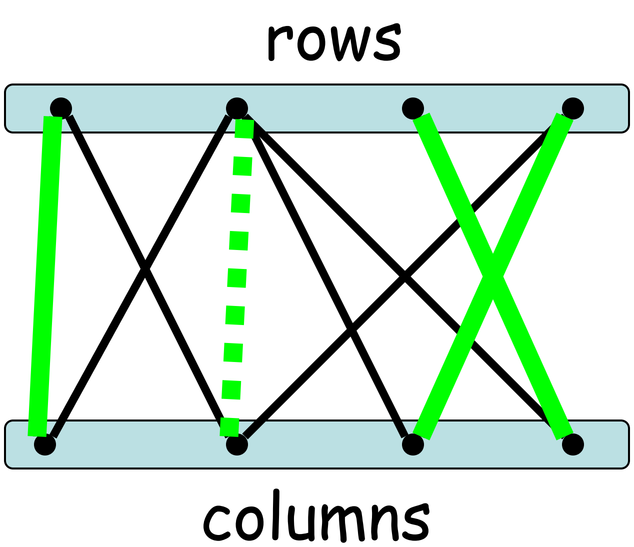

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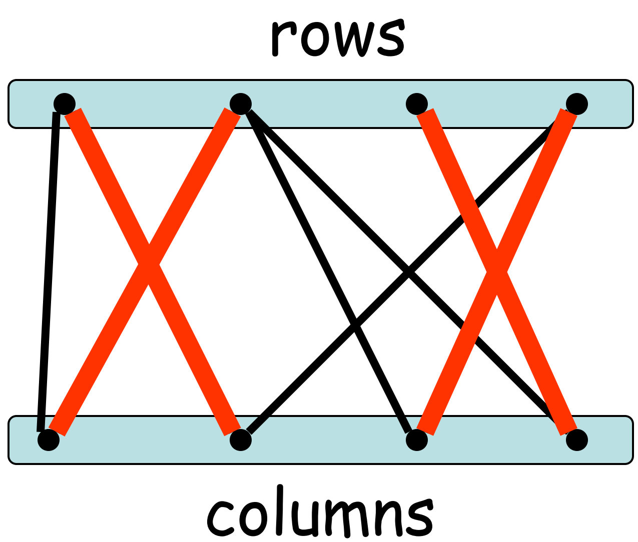


The permanent counts the number of perfect matchings.

Previous Work on the Permanent Problem

[Kasteleyn '67]

poly-time for planar graphs (bipartite or not)

[Valiant '79]

#P-complete for non-planar graphs

[Jerrum-Sinclair '89]

fpras for special graphs, e.g. the dense graphs,
based on a Markov chain by Broder '88

[Jerrum-Sinclair-Vigoda '01 & '05]

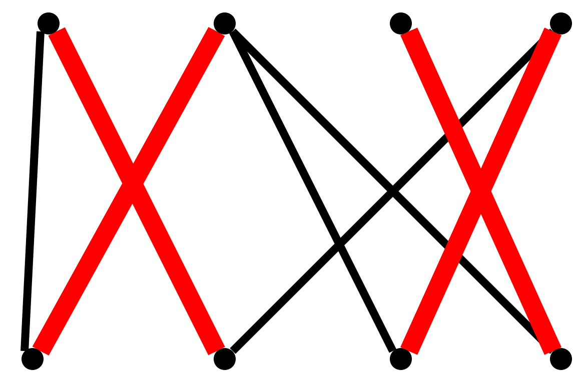
$O^*(n^{26})$ fpras for any bipartite graph, later $O^*(n^{10})$

Our result:

$\boxed{O^*(n^7)}$

Broder chain

uniform sampling of perfect matchings of a given graph



At a perfect matching:

- remove a random edge

At a near-matching:

- pick a vertex at random
  - if a hole, try to match with the other hole
  - otherwise slide (if can)

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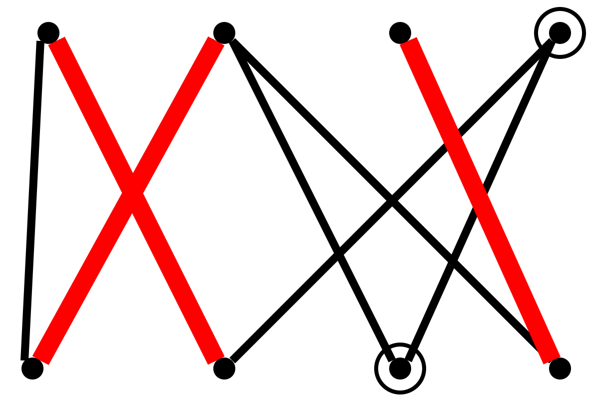

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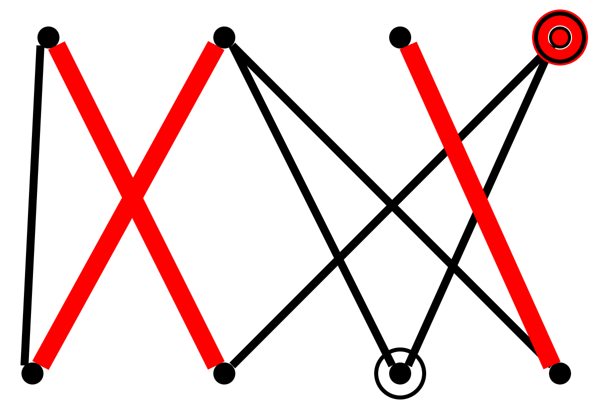

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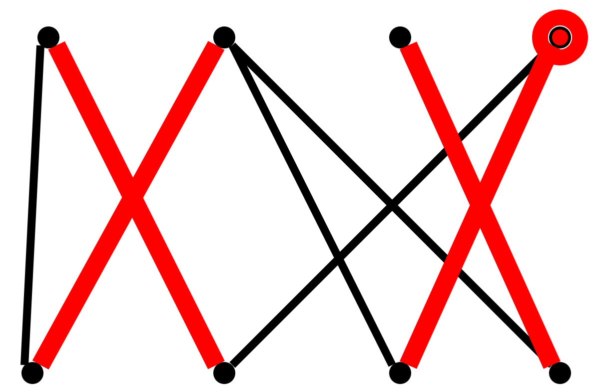

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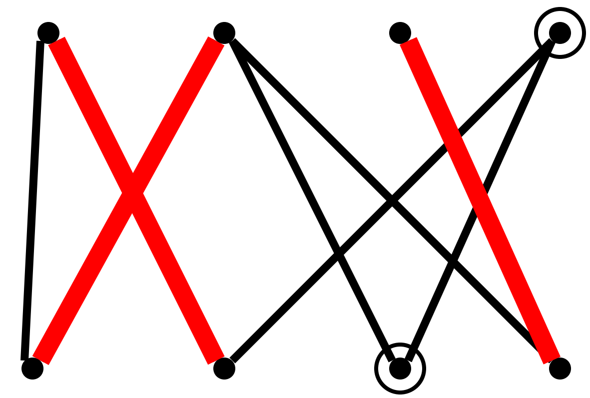

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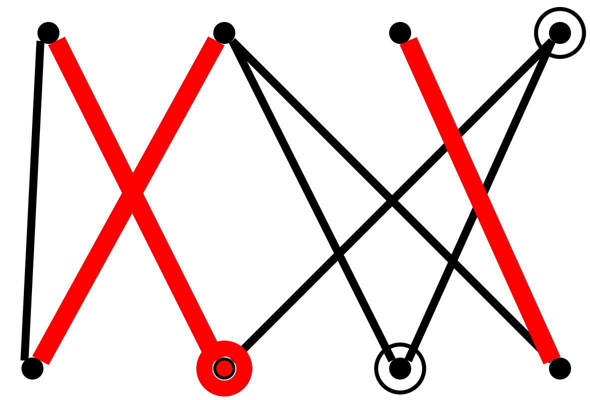

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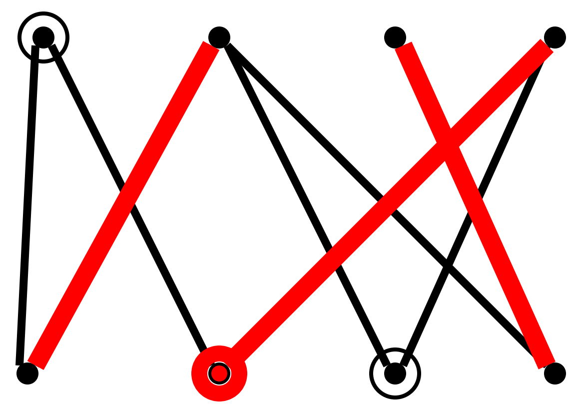

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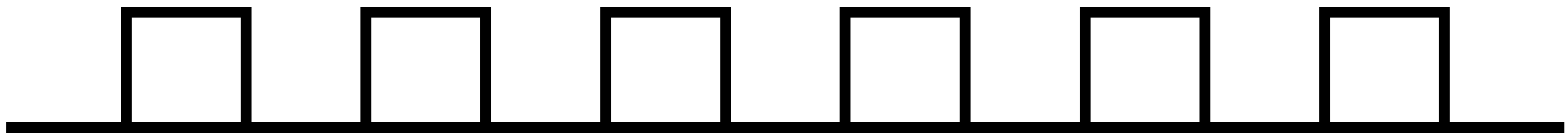
Does the Broder chain mix in polynomial time?



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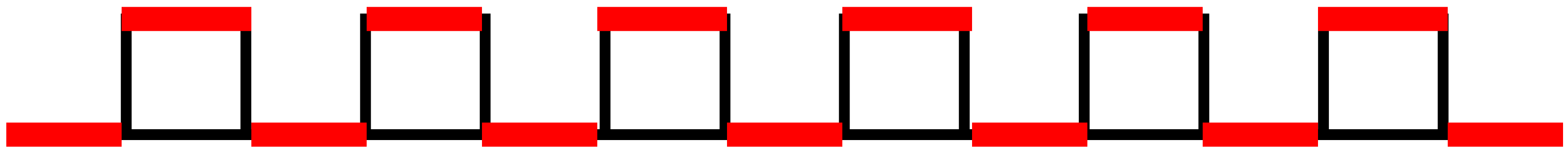

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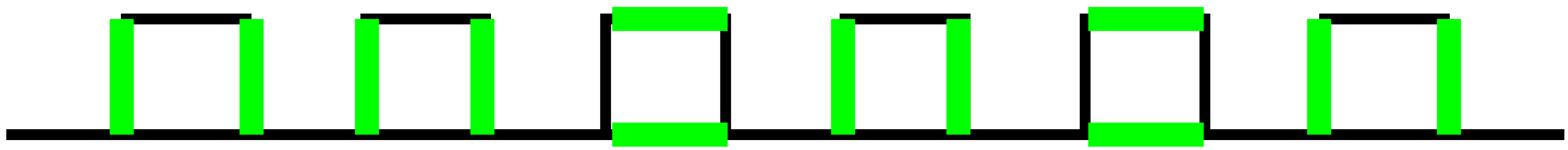

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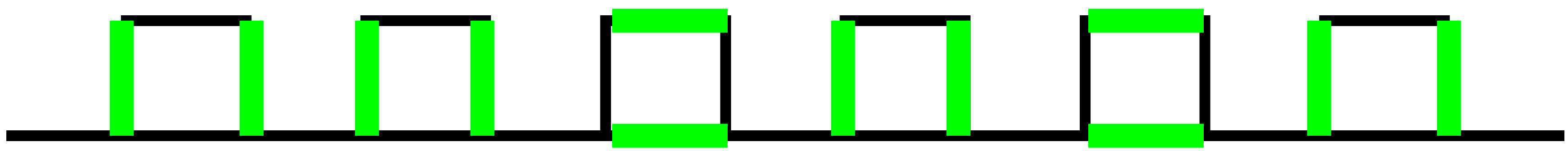
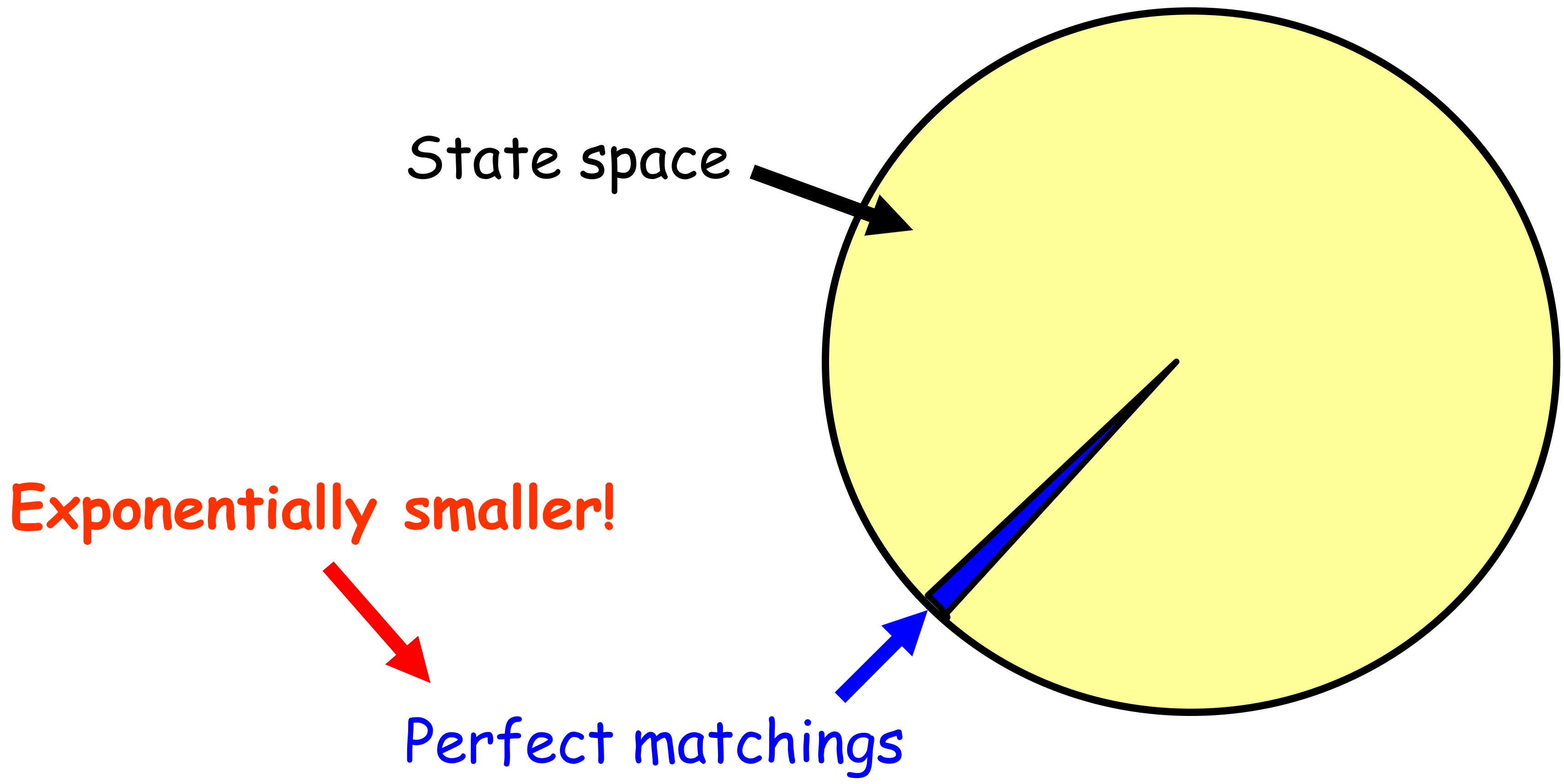
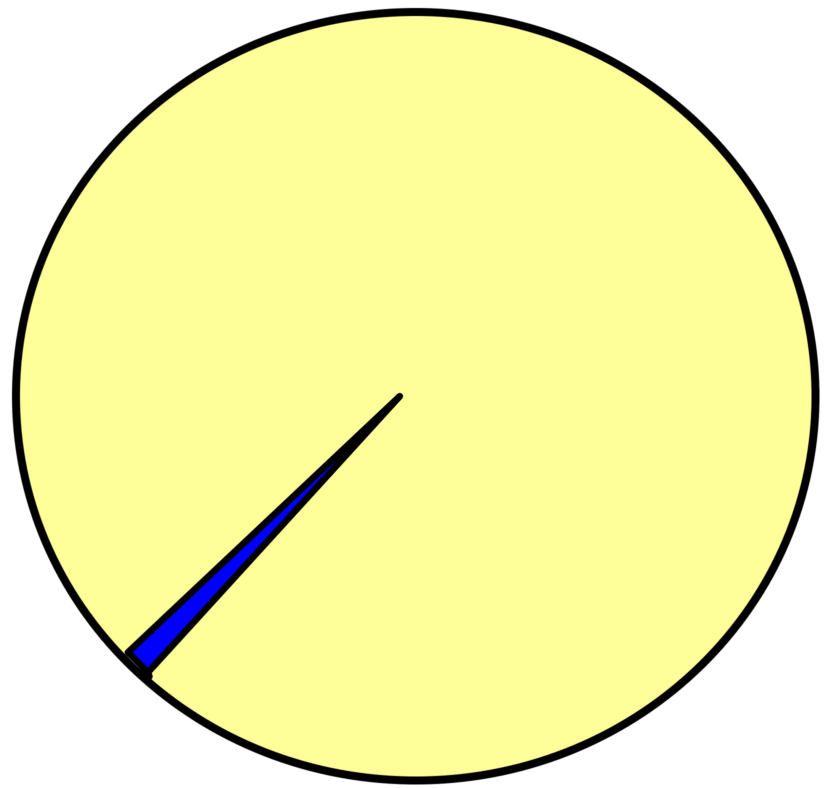


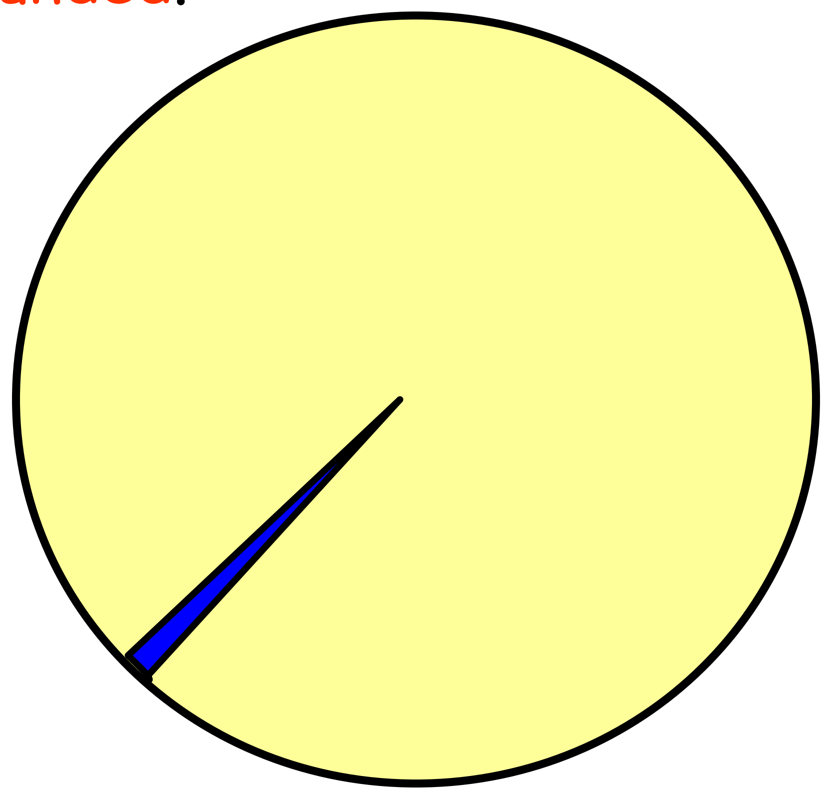


Theorem[JS]: Rapid mixing if perfect matchings polynomially related to near-perfect matchings.



Theorem[JS]: Rapid mixing if perfect matchings polynomially related to near-perfect matchings.

Idea[JSV]: Weight the states so that the weighted ratio is always polynomially bounded.



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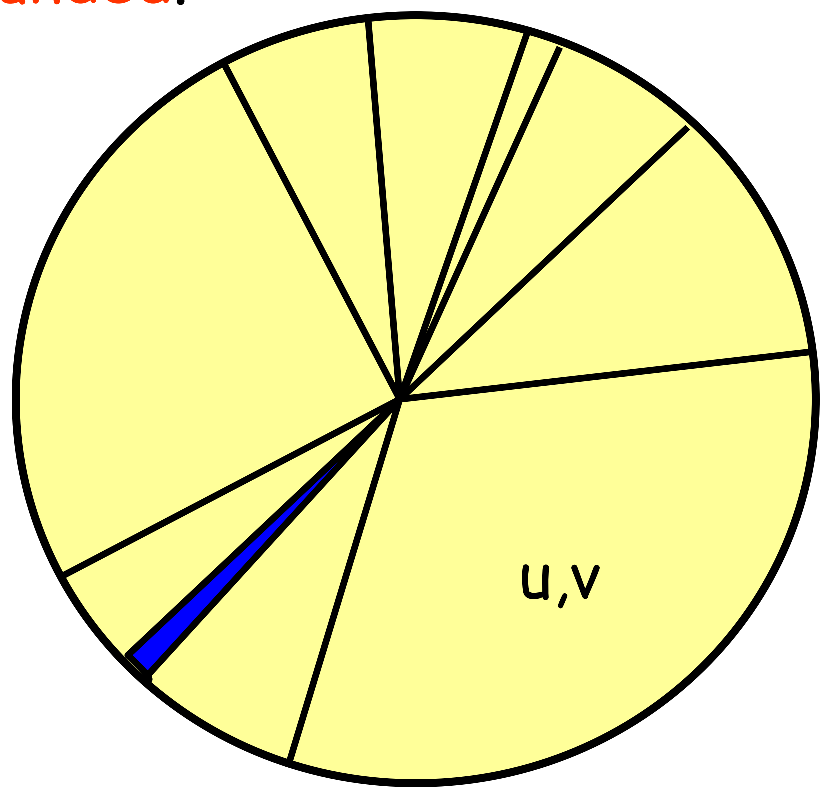
n^2+1 regions, very different weight



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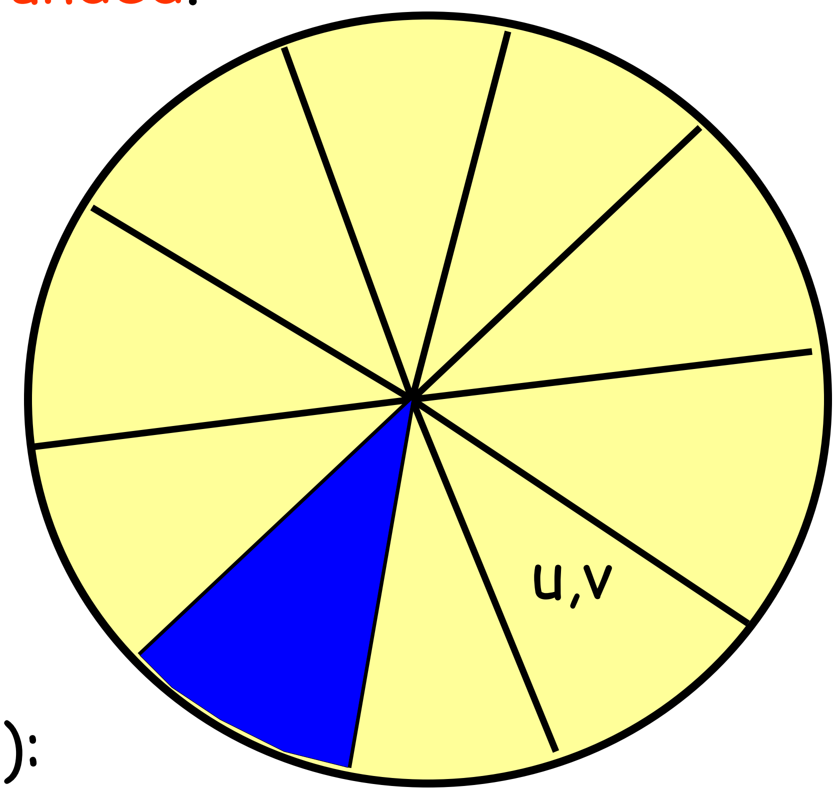
n^2+1 regions, each about the same weight

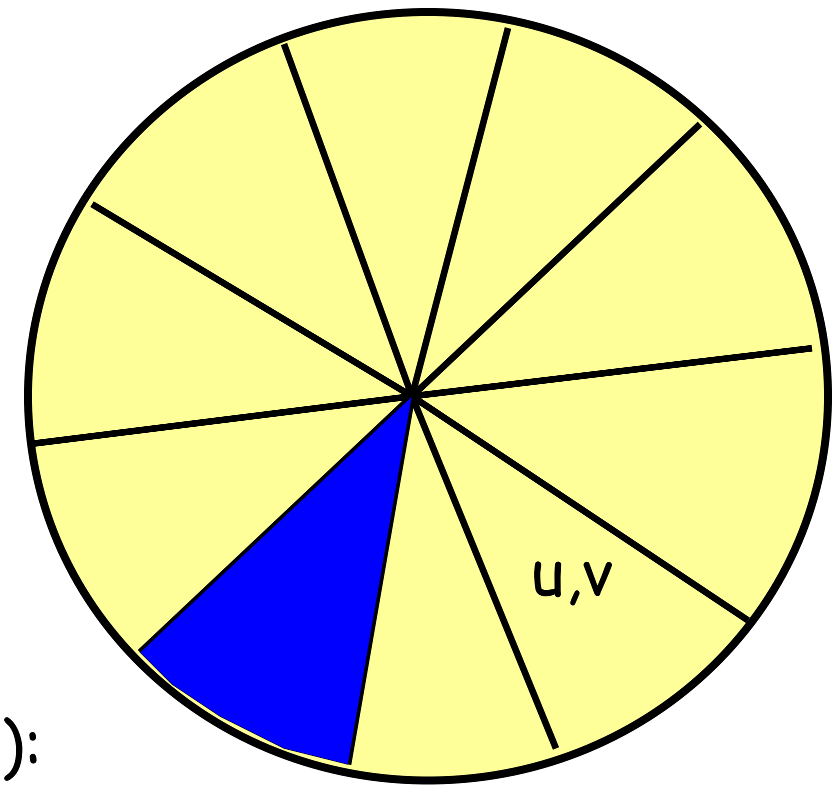


Ideal weights
(for a matching with holes u,v):

(# perfects) / (# nears with holes u,v)

Good: A perfect matching sampled with prob. $1/(n^2+1)$

Bad: Computing ideal weights as hard as original problem?



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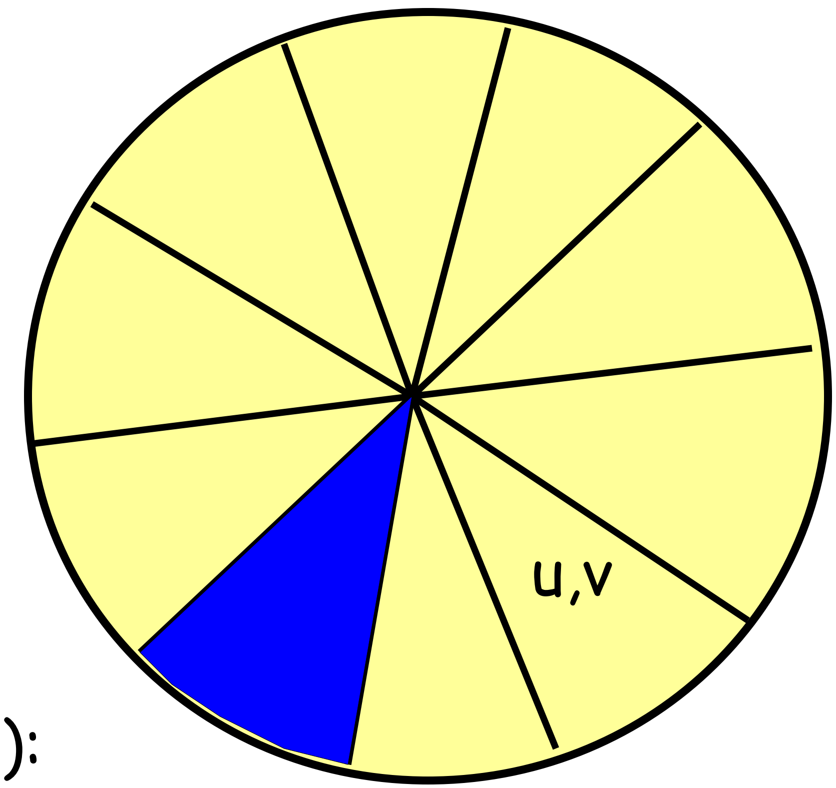
Solution: **Approximate** the ideal weights



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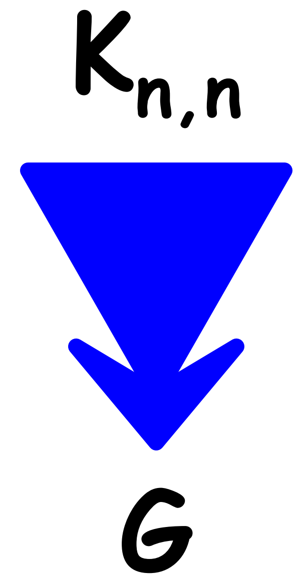
Simulated Annealing

Solution: **Approximate** the ideal weights



Start with an easy instance,

gradually get to the target instance.

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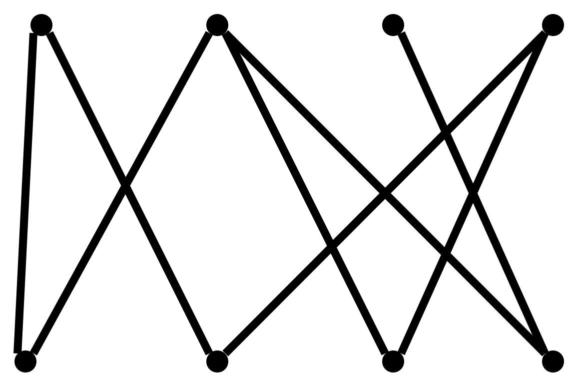

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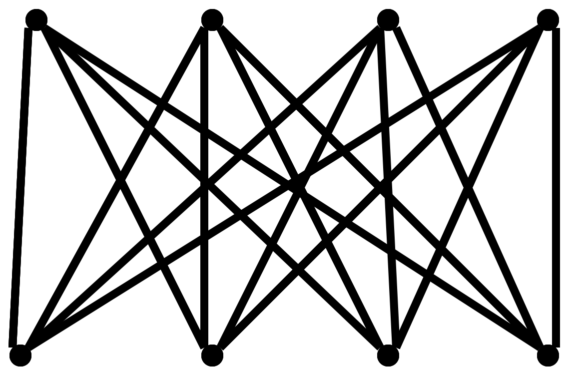

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- start with the complete graphs (weights easy to compute)

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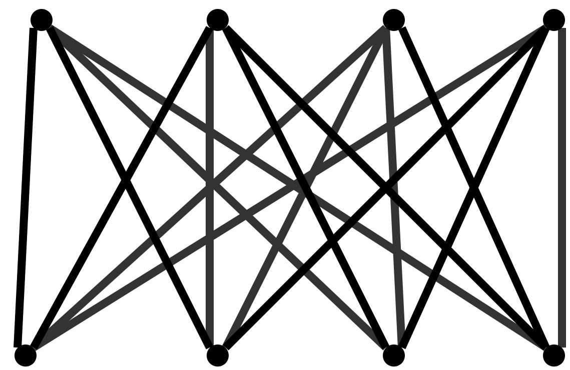

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- start with the complete graphs (weights easy to compute)
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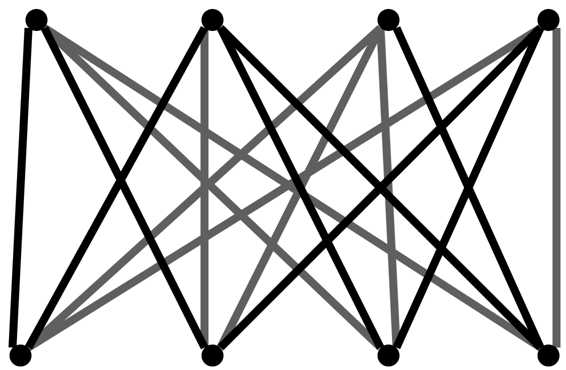

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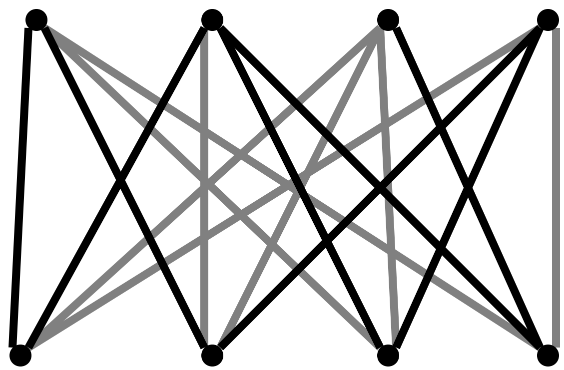

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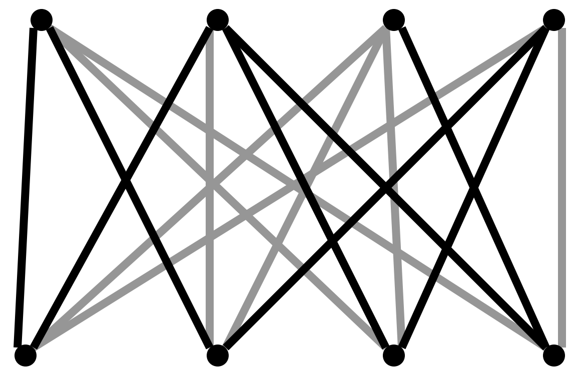

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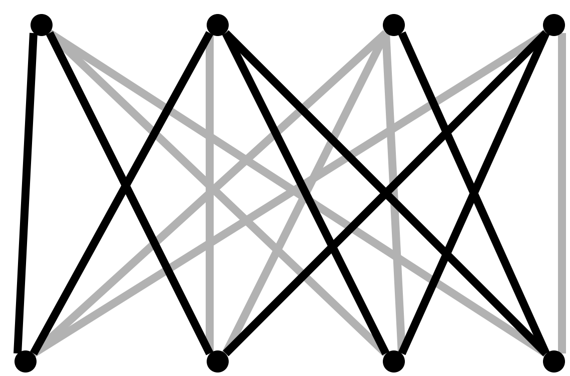

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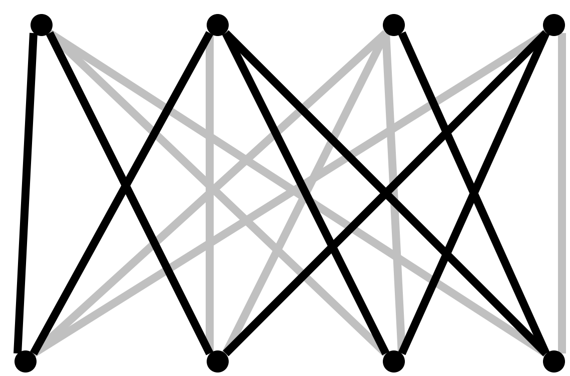

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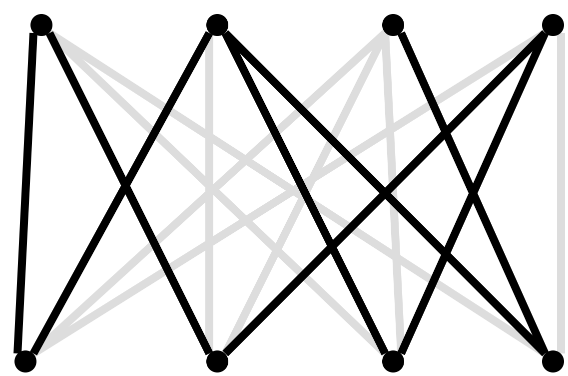

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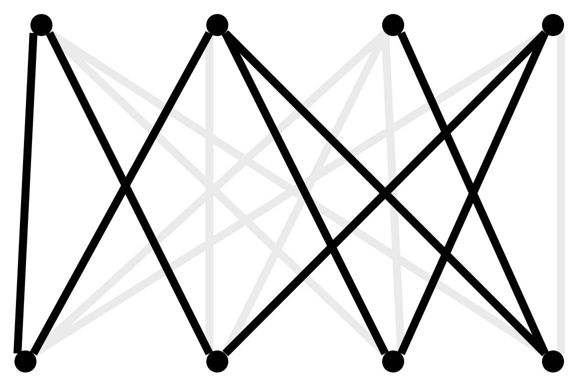

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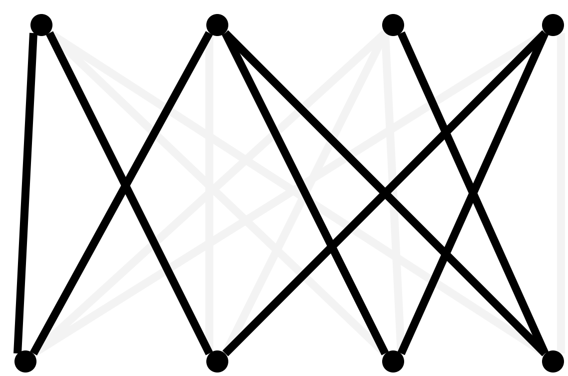

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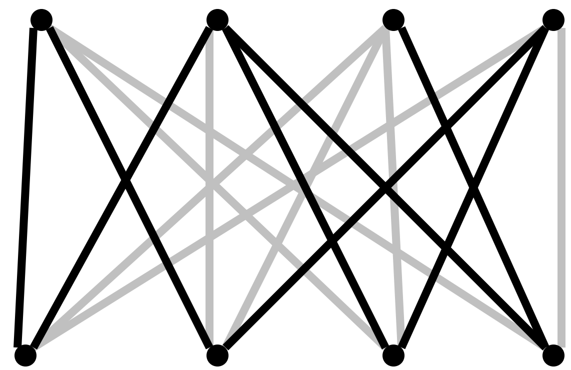
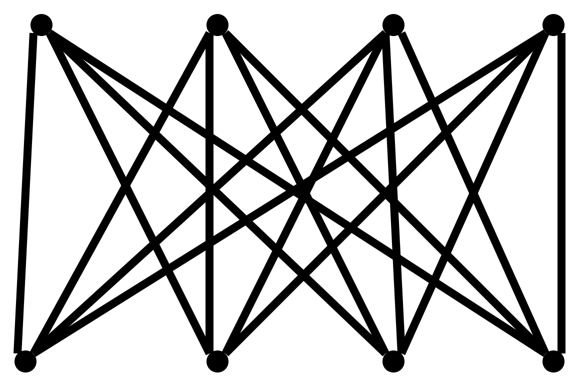


The edges have activities:

- 1 for a real edge
- $\lambda \in [0,1]$ for a non-edge

How activities help?

- start with $\lambda = 1$
- compute corresponding weights $n! \,/\, (n-1)!$



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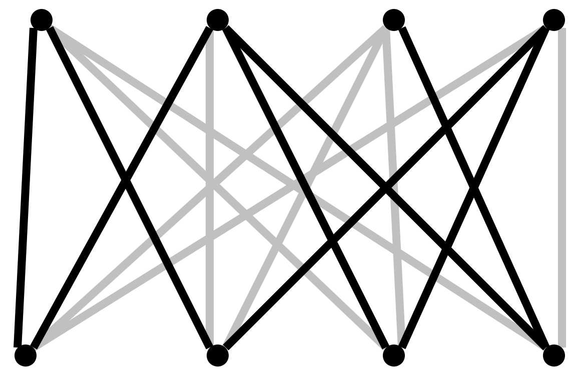
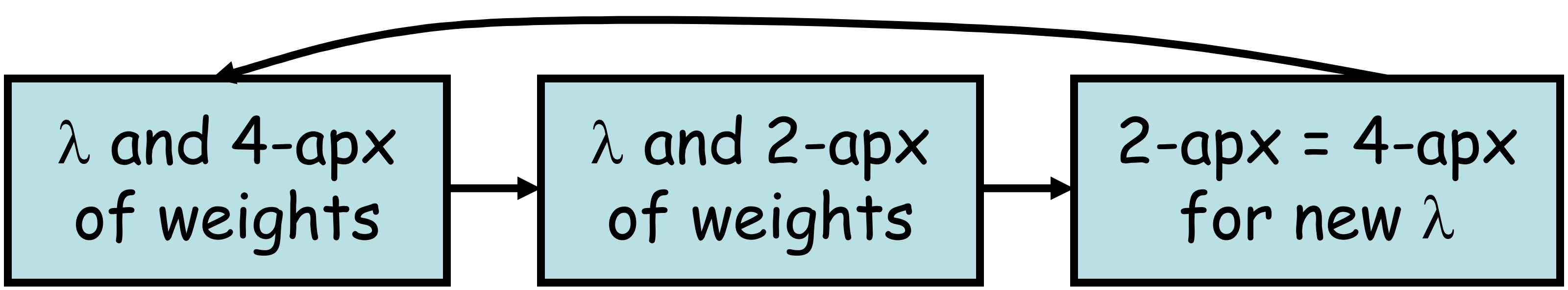


Repeat until $\lambda < 1/n!$



Running Time [JSV]

Thm: The (λ,hole-weights)-Broder chain mixes in time $O^*(n^6)$.

We need:

$O^*(n^6)$	per sample
$O^*(n^2)$	samples (boosting from 4-apx to 2-apx)
$O^*(n^2)$	λ-decrements (phases)
$O^*(n^{10})$	total to get a 2-apx of the ideal weights

Running Time [BŠVV]

Thm: The (λ,hole-weights)-Broder chain mixes in time $O^*(n^4)$ $\cancel{O^*(n^6)}$.

We need:

$O^*(n^4)$	$\cancel{O^*(n^6)}$	per sample
$O^*(n^2)$	$O^*(n^2)$	samples (boosting from 4-apx to 2-apx)
$O^*(n)$	$\cancel{O^*(n^2)}$	λ-decrements (phases)
$O^*(n^7)$	$\cancel{O^*(n^{10})}$	total to get a 2-apx of the ideal weights

Reformulation of the problem

Promise: a set of polynomials of degree n such that

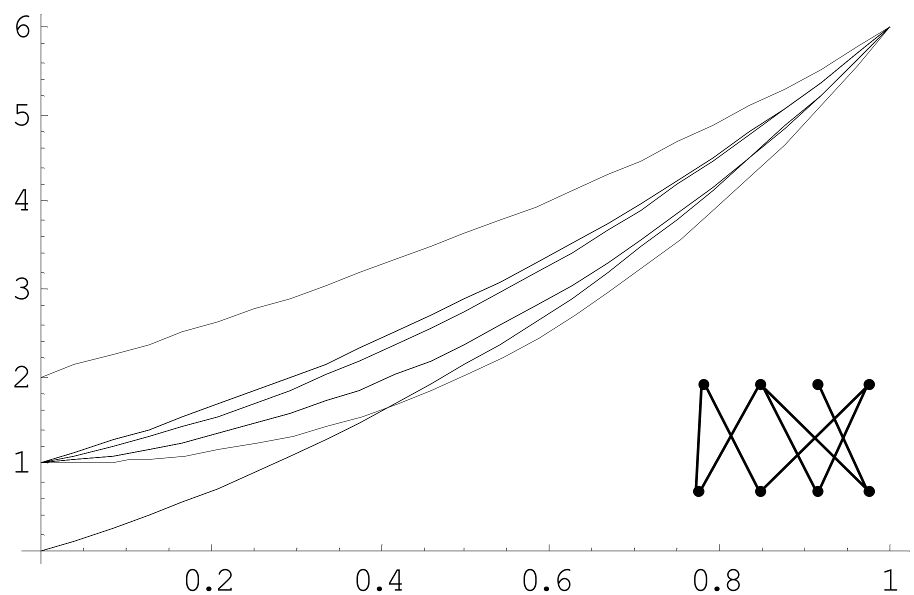
- polynomials have a low-degree term
- non-negative integer coefficients sum to $\leq$ n!

Goal: λ-sequence (from 1 to 1/n!) such that

for every polynomial ratio of consecutive values $\leq$ 2

Tricky part:

Don't know the coefficients!



Intuition

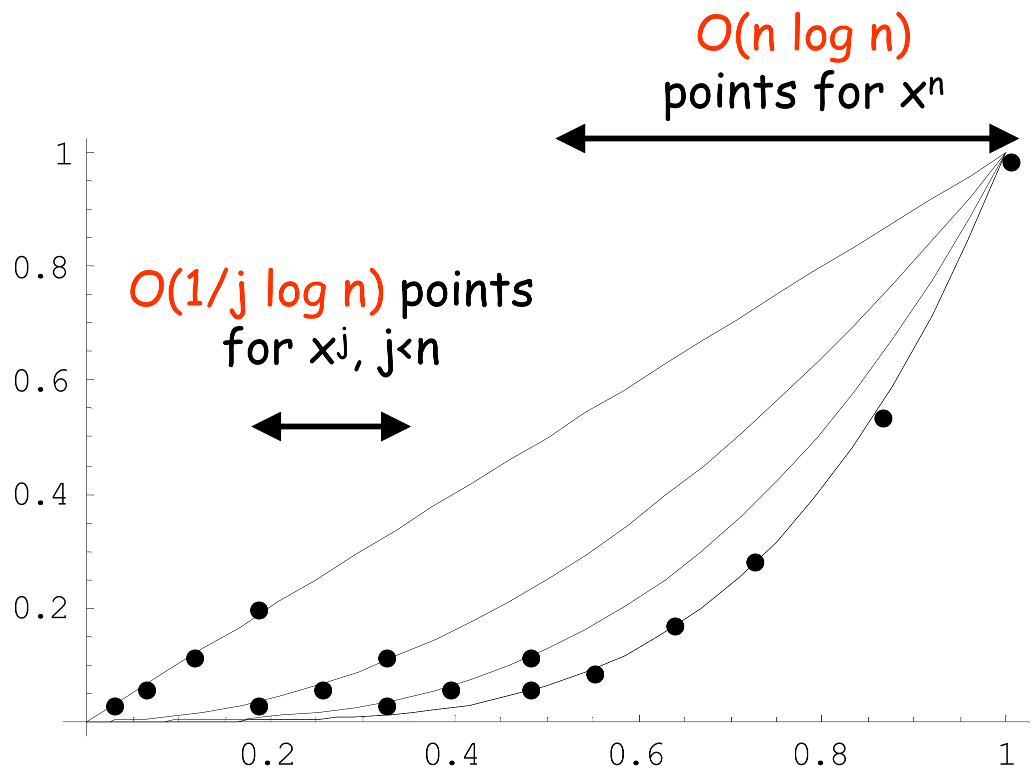
The worst case is the set of polynomials x^j, $j=1,\ldots,n$

Problem: no low-degree terms and x^n "dominates"

Fix: if the value of some x^j drops below $1/n!$, ignore the polynomial

TOTAL:

$\boxed{O(n \log^2 n)}$ points



Conclusions

- new cooling schedule: a blackbox, applicable to other problems
- improved analysis of the weighted Broder chain
- interest of practical comunity

Open Problems

- other applications of the cooling schedule
- faster mixing result
- do we need n^2 weights?
- non-bipartite graphs