On the Diaconis-Gangolli Markov Chain for Sampling Contingency Tables with Cell-Bounded Entries

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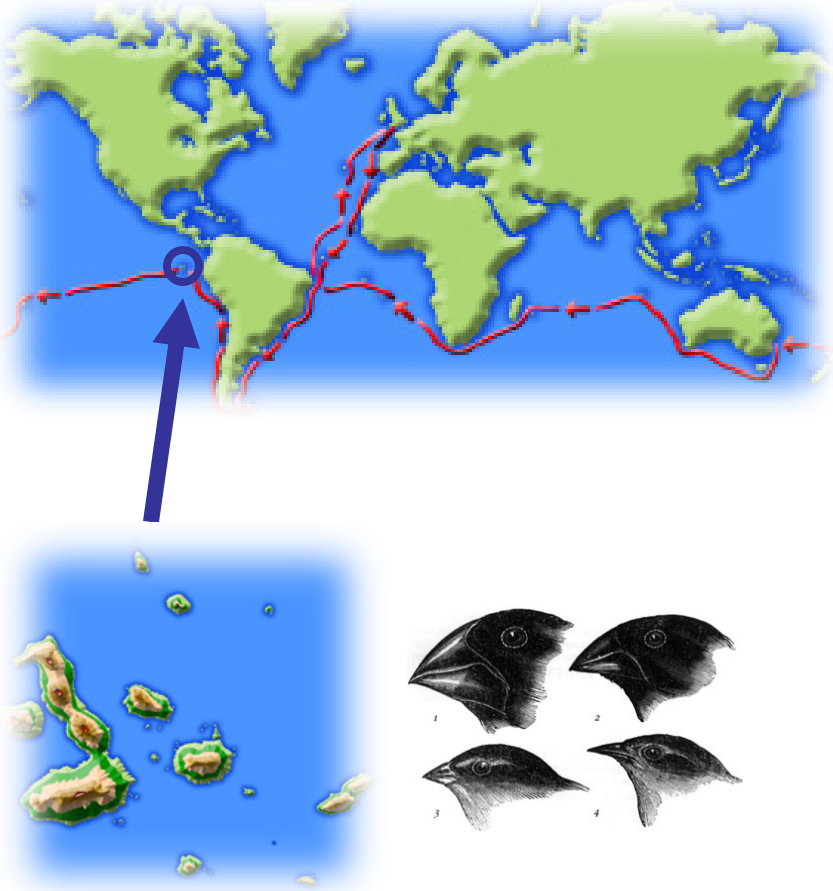
COCOON 2009, July 14, 2009

Parsing the title: Contingency Tables

A motivational example: Darwin's finches



1830's:

geological expedition around the world

1835:

Galápagos archipelago, Darwin observes that individual islands are inhabited by different types of finches

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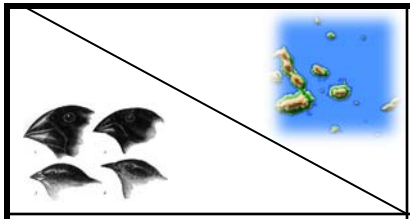	Santa Cruz	Plaza	Santa Fe	San Cristobal	Espanola	Floreana	Isabela	Fernandina	Santiago	Rabida	Genovesa	
Small ground finch	1	1	1	1	1	1	1	1	1	1		10
Medium ground finch	1	1	1	1		1	1	1	1	1		9
Large ground finch	1						1	1	1	1	1	6
Cactus ground finch	1	1	1	1		1	1		1	1		8
Large cactus ground finch					1						1	2
Sharp-beaked ground finch								1	1		1	3
Vegetarian finch	1			1		1	1	1	1	1		7
Small tree finch	1		1	1		1	1	1	1	1		8
Medium tree finch						1						1
Large tree finch	1					1	1	1	1	1		6
Woodpecker finch	1			1			1		1			4
Mangrove finch							1	1				2
Warbler finch	1		1	1	1	1	1	1	1	1	1	10
	9	3	5	7	3	8	10	9	10	8	3	

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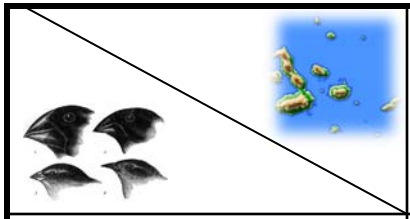	Santa Cruz	Plaza	Santa Fe	San Cristobal	Espanola	Floreana	Isabela	Fernandina	Santiago	Rabida	Genovesa	
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chance

OR

competitive pressures

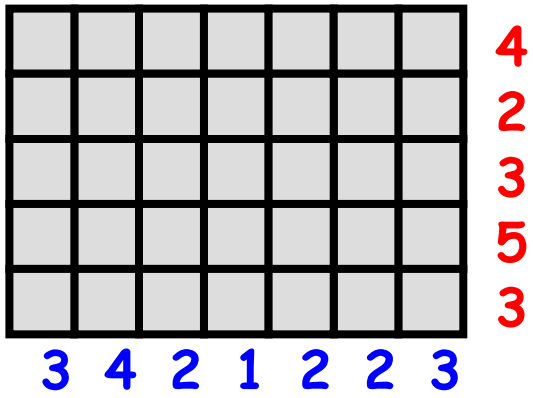
?

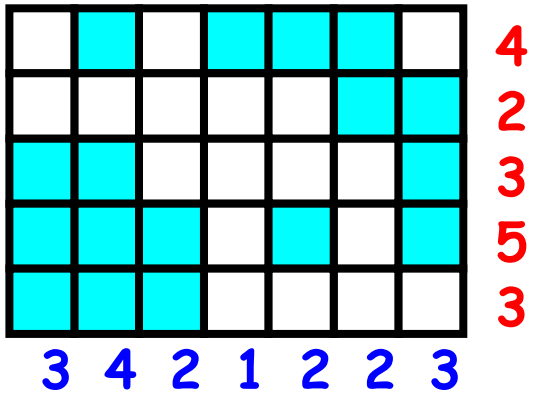
Parsing the title: Contingency Tables

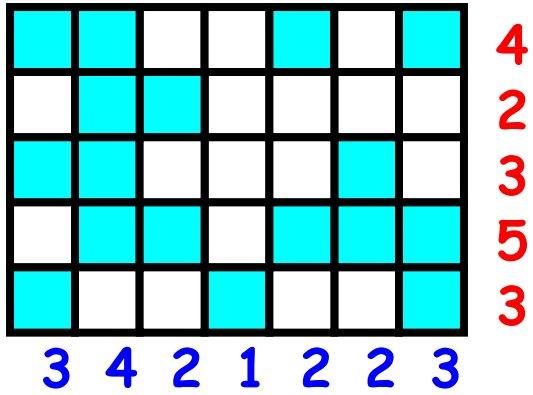
Formal definition:

Input: marginals (row/column sums)
Goal: obtain a uniformly random (binary or not) table that satisfies the given marginals

For example:
- input on the left, two possible binary tables on the right







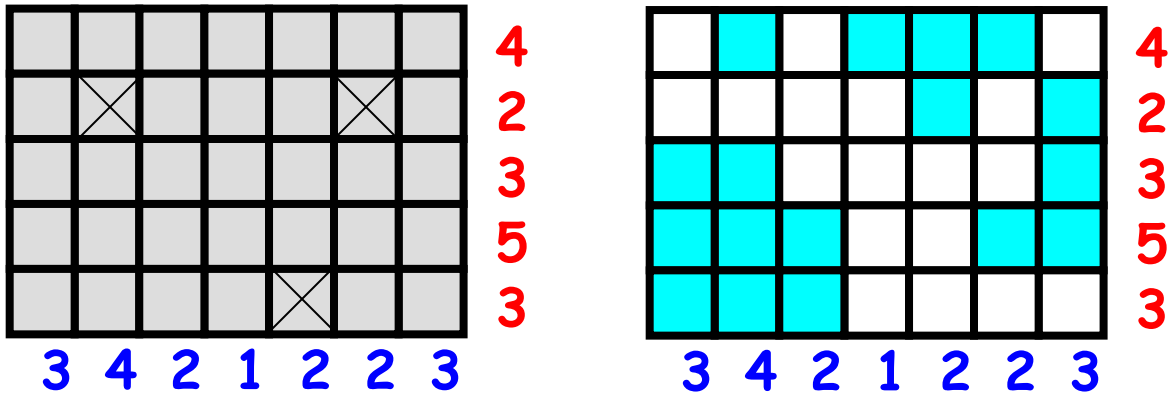
Parsing the title: Cell-bounded Cont.Tables

Formal definition, generalized:

Input: marginals (row/column sums) and cell-bounds (an upper-bound for every cell)

Goal: obtain a uniformly random table that satisfies the given marginals and cell-bounds

For example:
- input on the left (cell-bounds are 1, crossed cells 0), a possible table on the right



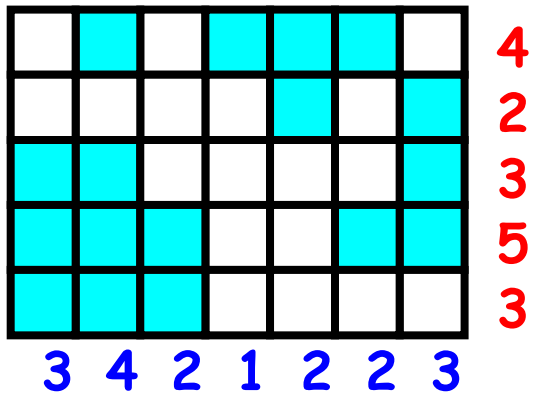
Parsing the title: Diagonis-Gangolli MC

Diaconis-Gangolli's Markov chain ['94]:

- start at some valid table
- then in each step of the MC:
  - choose a random 2x2 subtable and one of its (two) diagonals
  - try to decrease both entries by 1 and increase the other entries by 1:
    if within cell-bounds, keep the changes, otherwise ignore the changes

Example:

All cell-bounds are 1.



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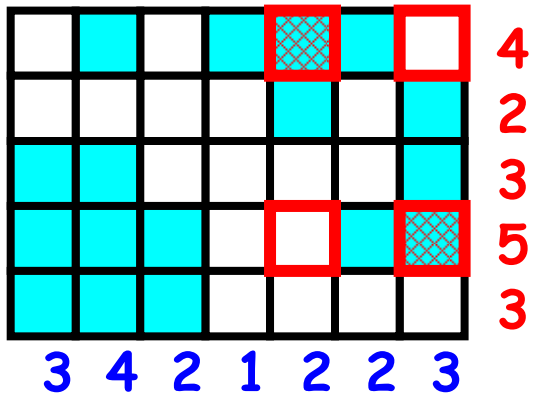

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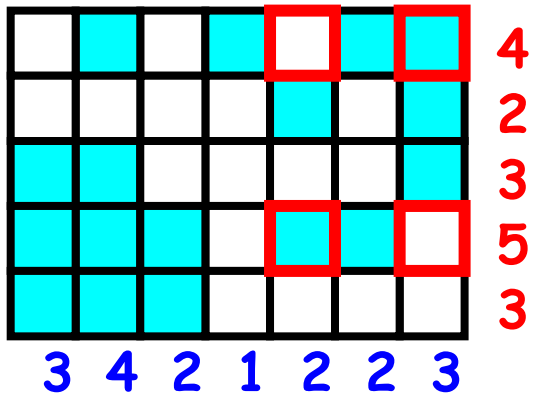

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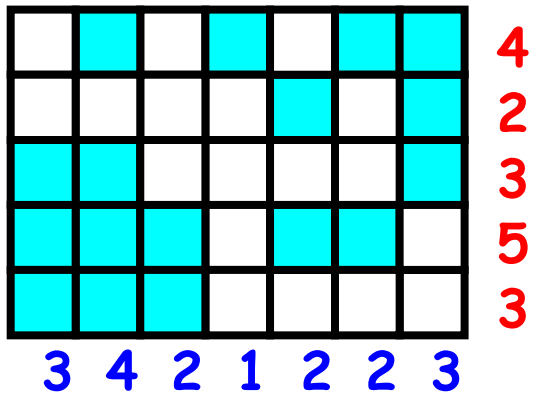

OK

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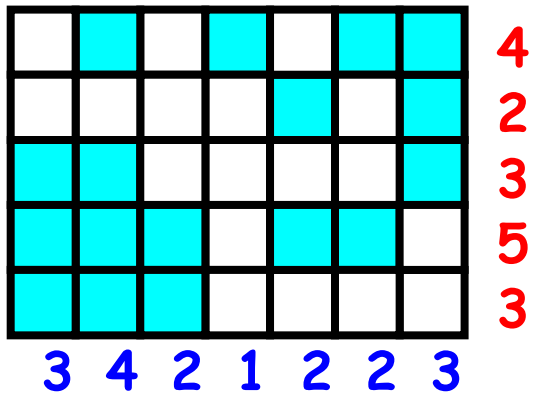
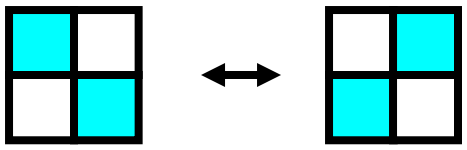


Schematically for cell-bounds 1:



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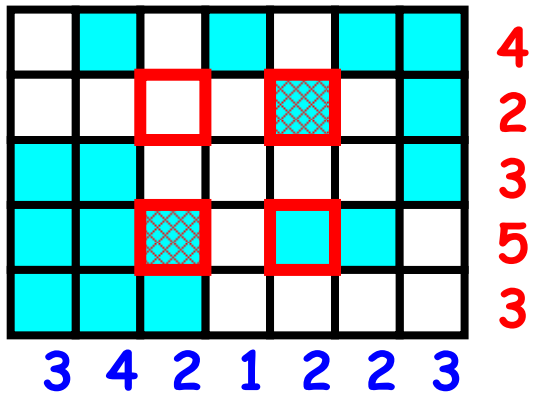

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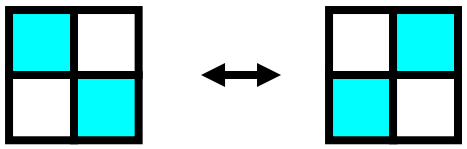

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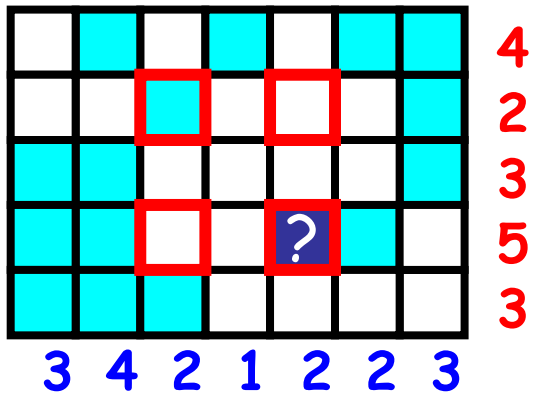

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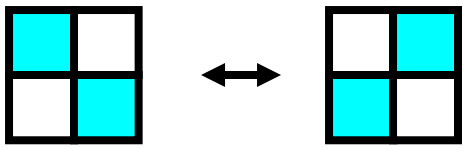

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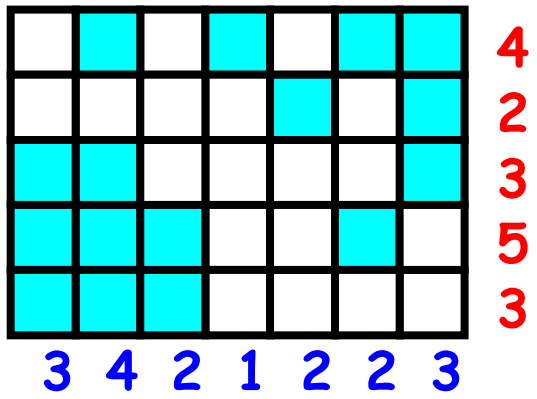

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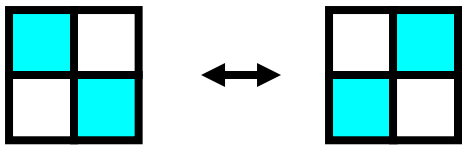


Parsing the title: Diagonis-Gangolli MC

Remarks:

- the original chain was proposed for contingency tables (without cell-bounds)

- without cell-bounds, or with all cell-bounds 1:
the chain connects the state space (i.e., can move from every table to every other table) and the stationary distribution is uniform

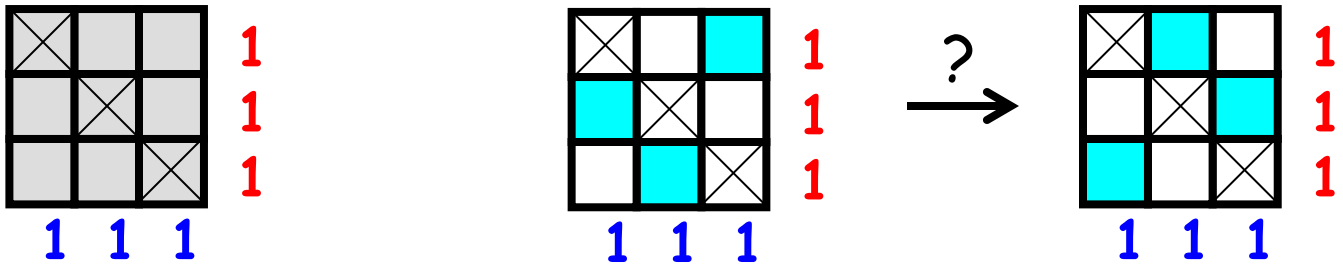
- OPEN: does the chain mix rapidly ? (i.e., does polynomial number of steps suffice to get a random table ?)

Parsing the title: Diagonis-Gangolli MC

OPEN QUESTION: does the chain mix rapidly ?

With cell-bounds:

- the state space might not be connected, e.g.:



- if the state space is connected, does the chain mix rapidly?

  This work: the answer is NO.

Remark: why cell-bounds ? One of the reasons: self-reducibility.

Related Work

Polynomial-time algorithms for:

- no cell-bounds, large marginals
[Dyer-Kannan-Mount '97, Morris '02]

- no cell-bounds, constant number of rows
[Cryan-Dyer '03, Cryan-Dyer-Goldberg-Jerrum-Martin '02, Dyer '03]

- cell-bounds, large marginals or constant number of rows
[Cryan-Dyer-Randall '05]

- all cell-bounds 1, (near-)regular marginals: rapid mixing of the Diaconis-Gangolli chain
[Kannan-Tetali-Vempala '99]

- all cell-bounds 1
[reduction from permanent: Jerrum-Sinclair-Vigoda '04,
directly: Bezáková-Bhatnagar-Vigoda '07]

Related Work

Polynomial-time algorithms for:

This work:

cell-bounds, slow mixing

- no cell-bounds, large marginals
[Dyer-Kannan-Mount '97, Morris '02]

- no cell-bounds, constant number of rows
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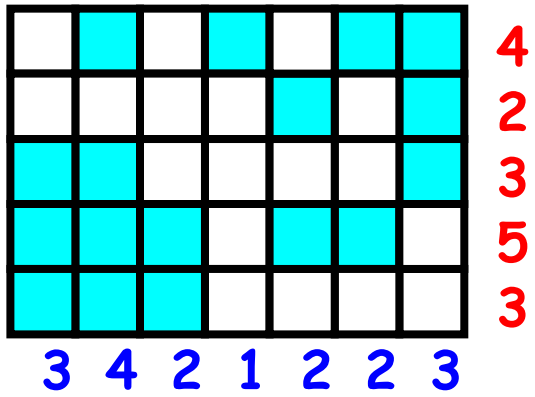
Graphs with Given Degree Sequence

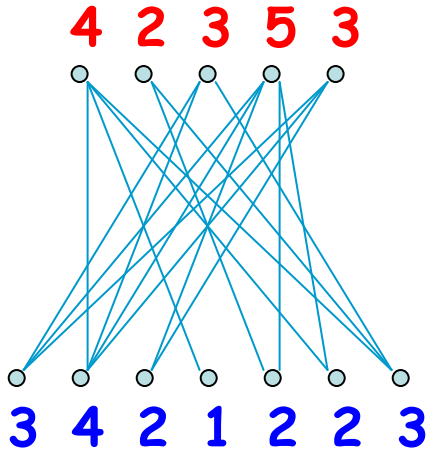
Binary contingency tables = a bipartite graph problem:

Input: degrees of all vertices

Goal: sample all graphs satisfying this degree sequence

Example:





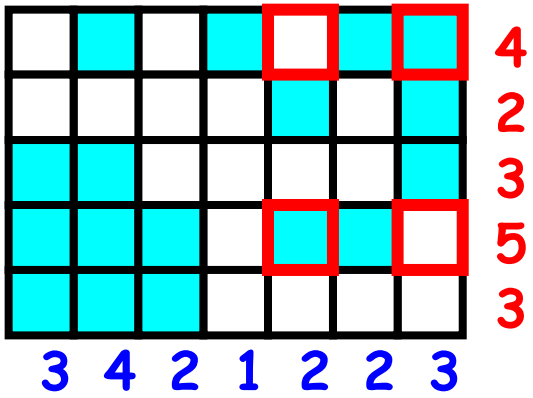
Note: cell-bounds = forbidden edges

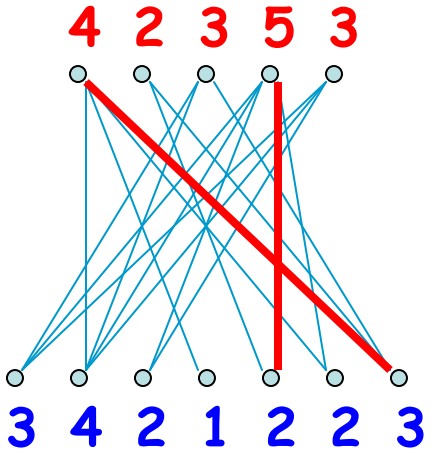
Graphs with Given Degree Sequence

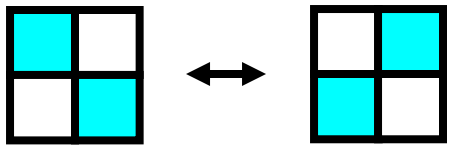
The Diaconis-Gangolli move on graphs:

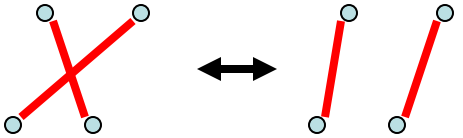
- choose two edges on random, re-match the end-points (if possible, i.e., if the new edges-to-be do not already exist)

Example:









Negative Result for Dense Graphs

Thm: There exists a family of inputs where

a) the chain connects the state space,

b) the number of allowed edges incident to every vertex is at least n/4,

c) the mixing time is at least exponential.

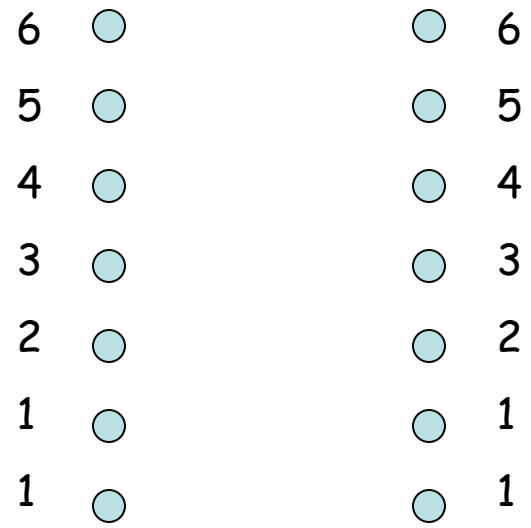
Why are dense inputs interesting ?

- non-cell-bounded tables are dense
- other sampling problems behave well for dense inputs

Negative Result for Dense Graphs

Thm: Exponential mixing time for instances with n/4-dense allowed edges.

Proof idea: consider this input (no forbidden edges)



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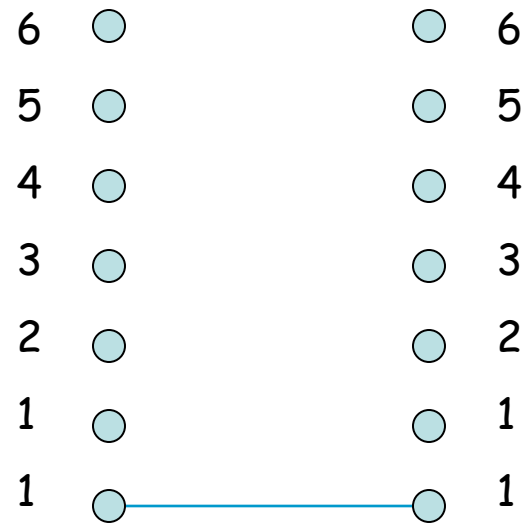

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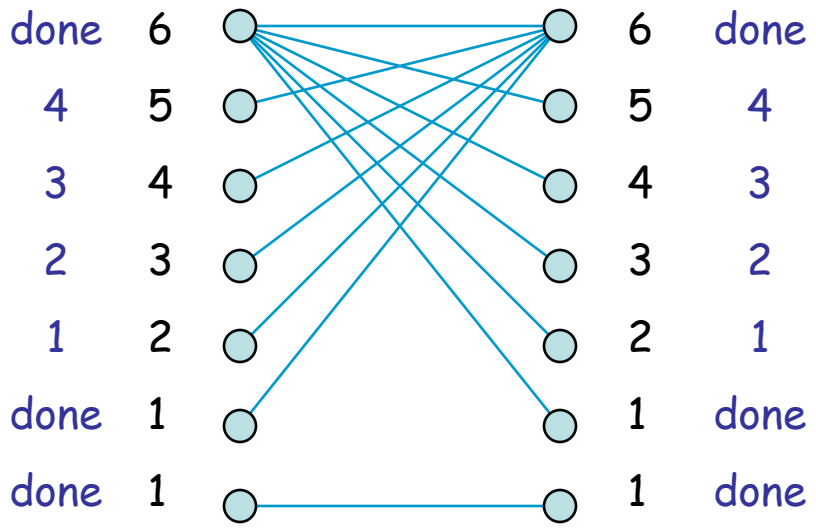

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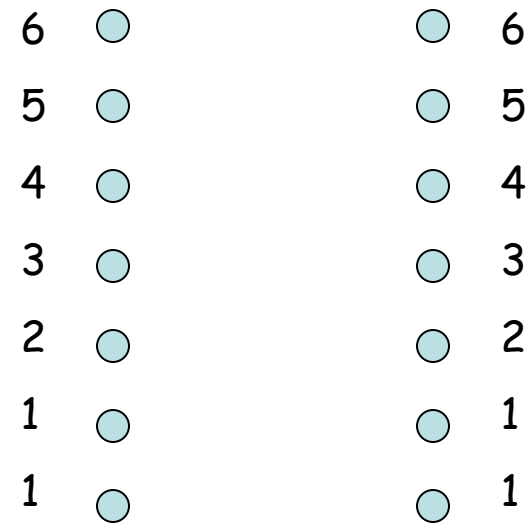

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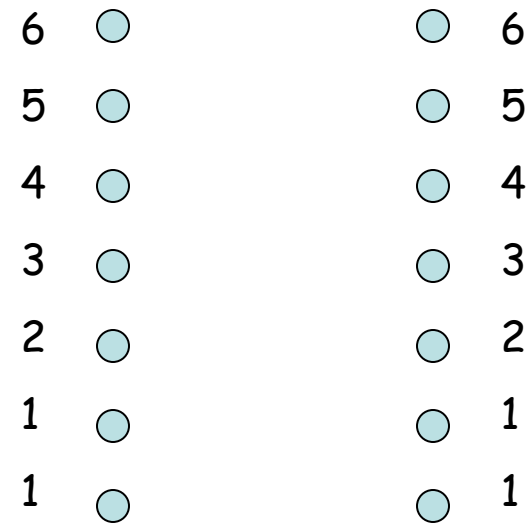
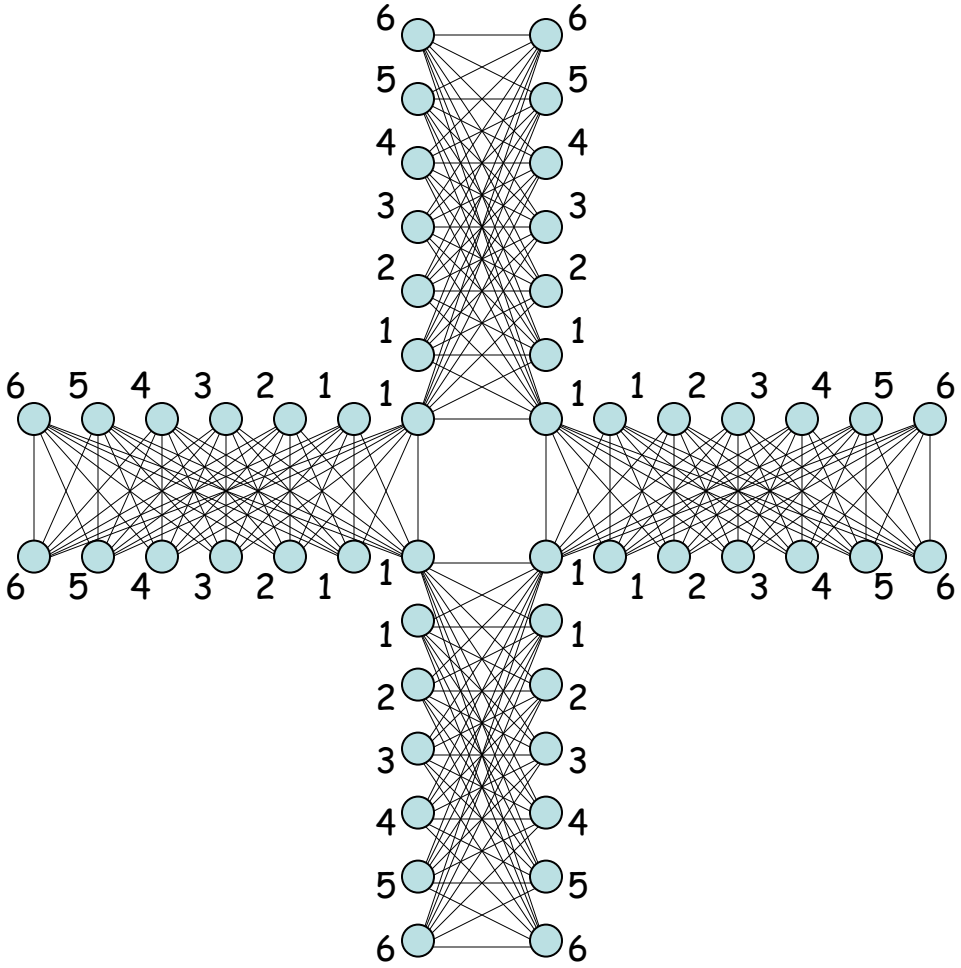


Negative Result for Dense Graphs

Thm: Exponential mixing time for instances with n/4-dense allowed edges.

Proof idea:

Let's connect four copies, the shown edges are the only allowed edges (i.e., the cell-bound for the corresponding cells is 1):



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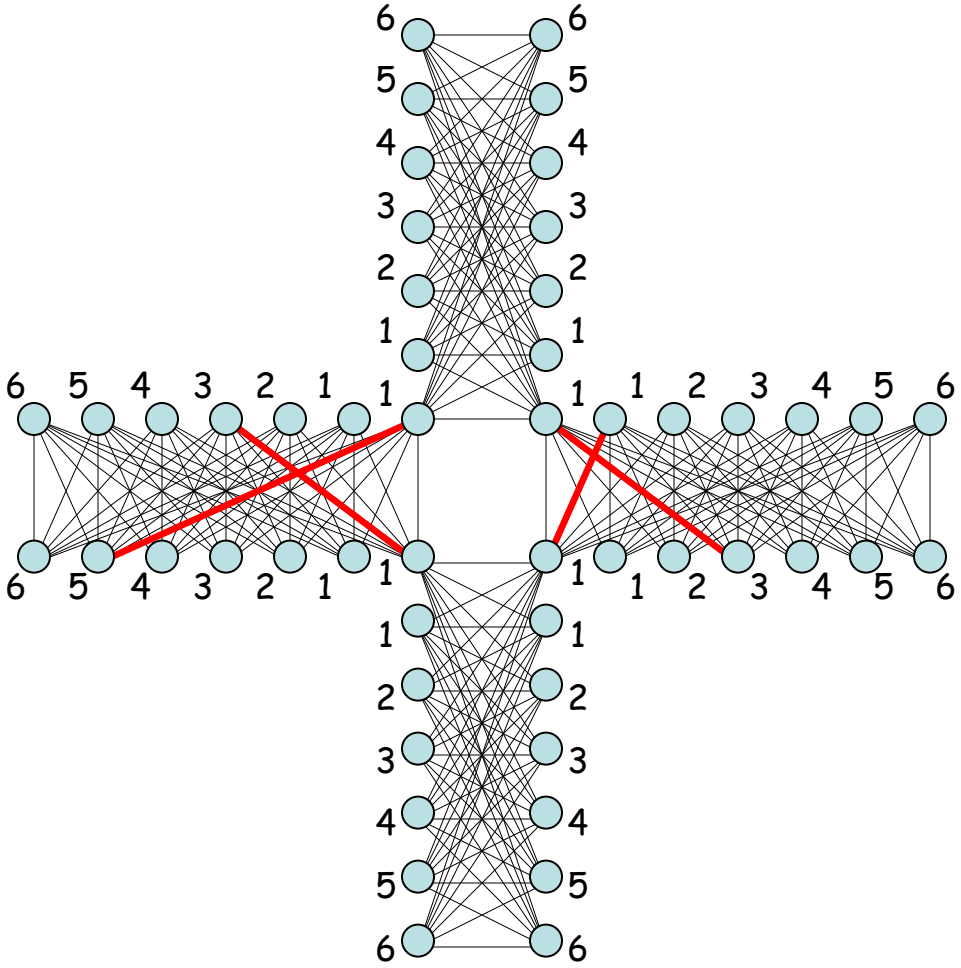
Proof idea:

Consider a graph with the required degree sequence, suppose the middle vertices connected as shown:



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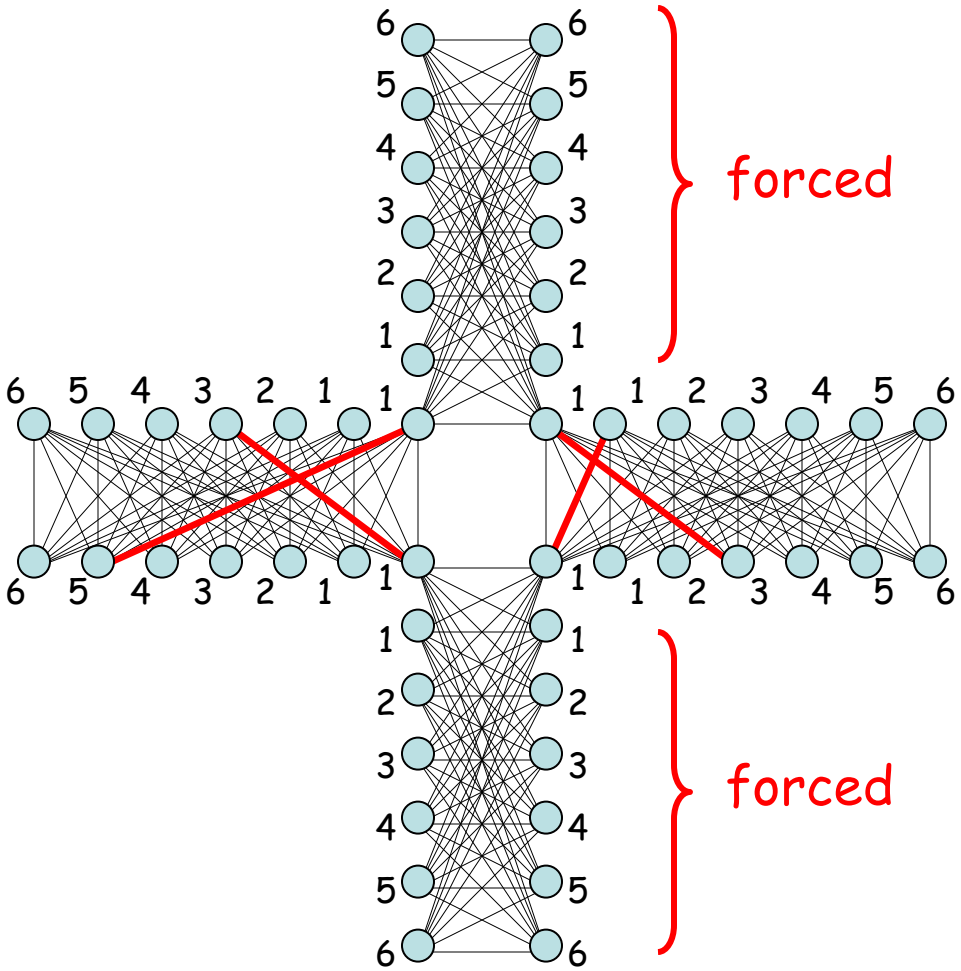

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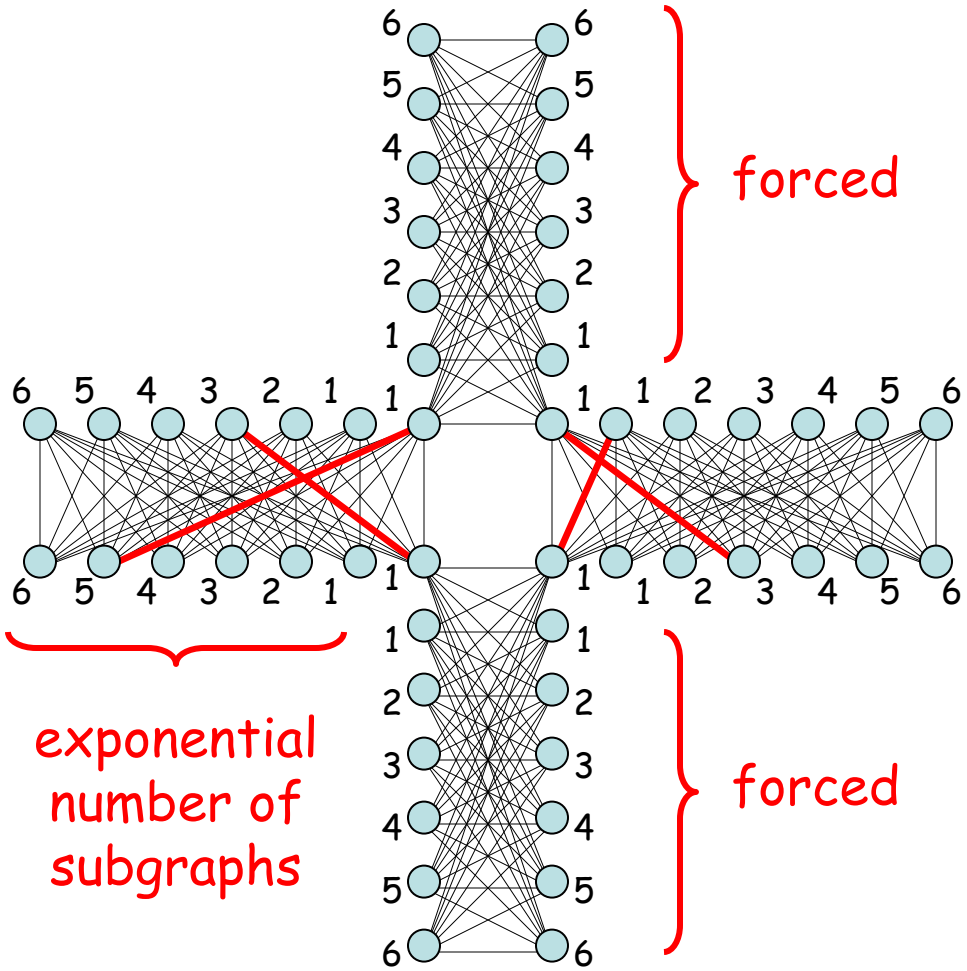

Conclusion: exponential number of "horizontal" possibilities.

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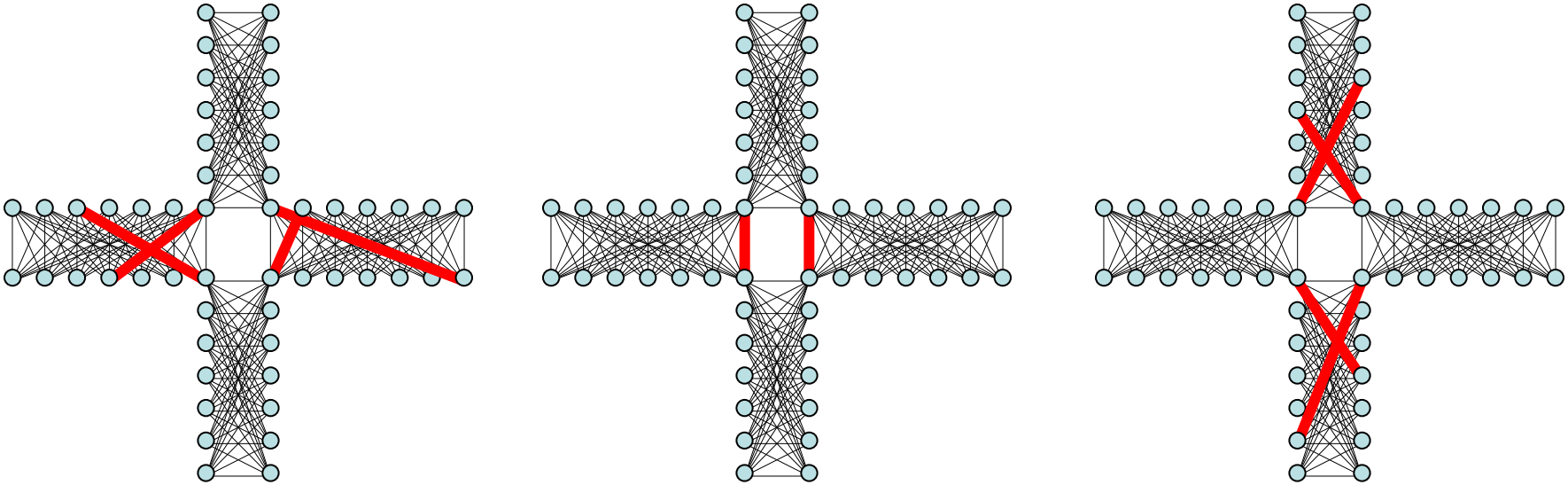
Thm: Exponential mixing time for instances with n/4-dense allowed edges.

Proof idea:

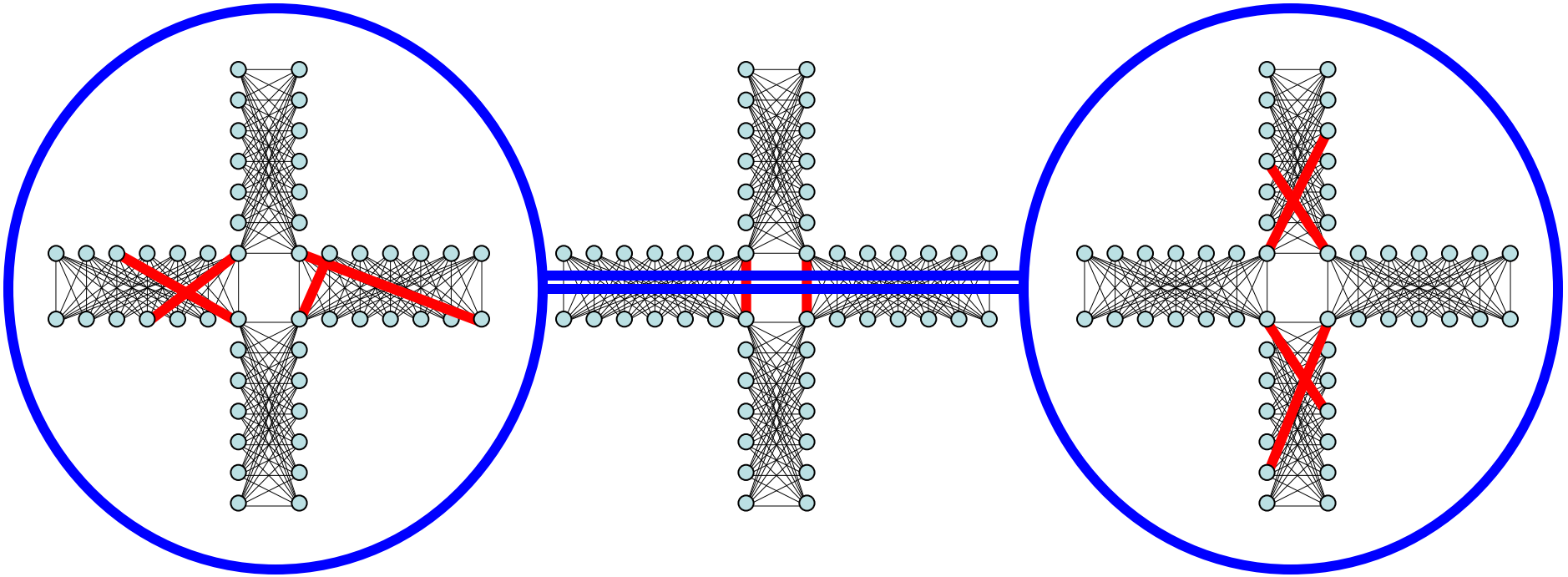


exponential one exponential

Negative Result for Dense Graphs

Thm: Exponential mixing time for instances with n/4-dense allowed edges.

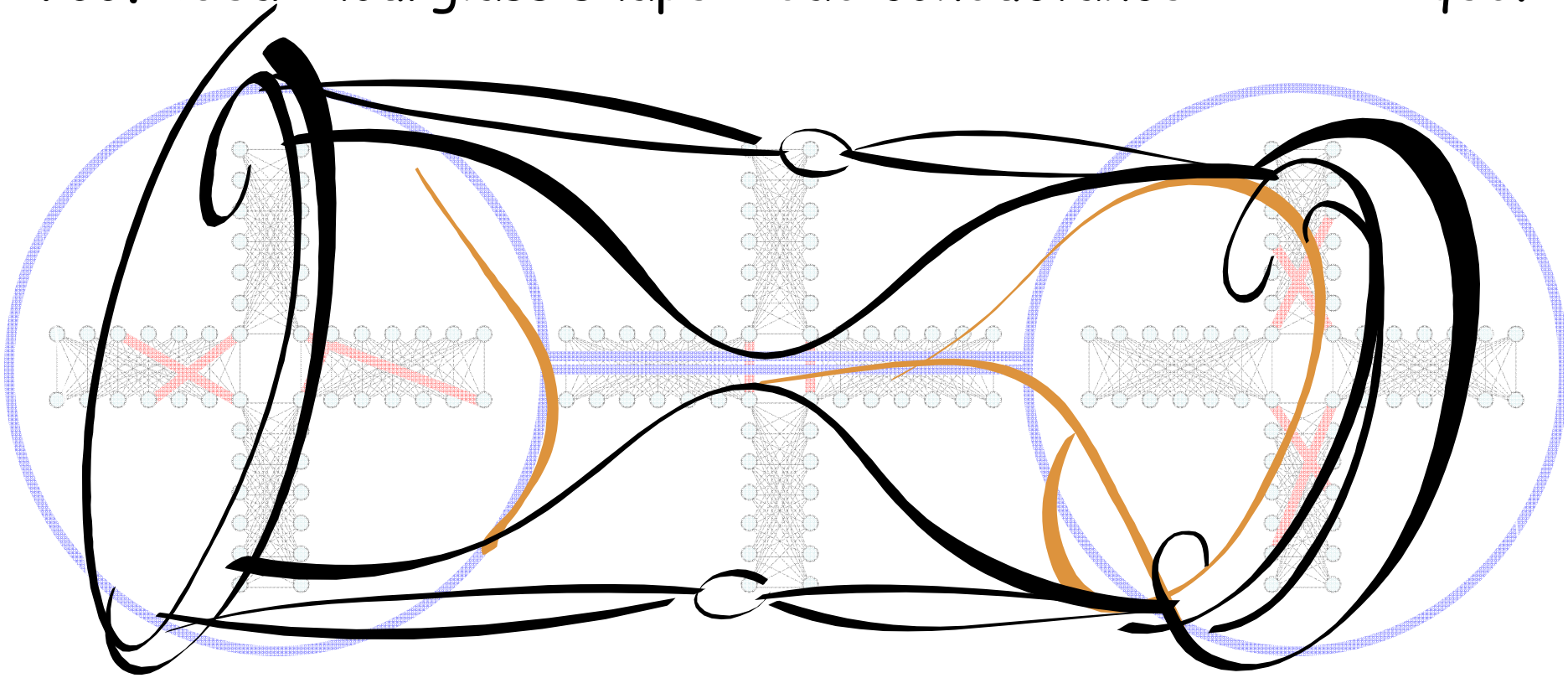
Proof idea:



Negative Result for Dense Graphs

Thm: Exponential mixing time for instances with n/4-dense allowed edges.

Proof idea: hourglass shape = bad conductance qed.



Negative Result for Regular Inputs

Thm: There exists a family of inputs where

a) the chain connects the state space,

b) all required degrees (i.e., the marginals) are equal,

c) the mixing time is at least exponential.

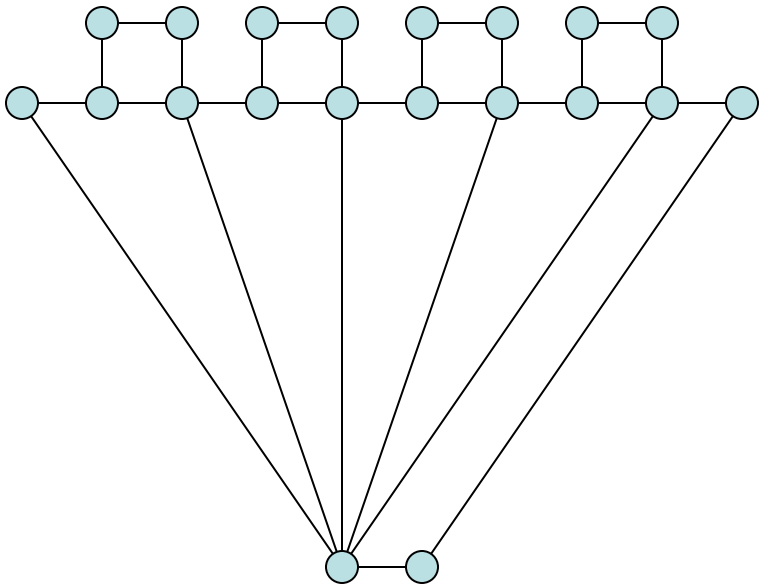
Why are regular inputs interesting ?

- if all cell-bounds are 1, the chain is known to mix rapidly for regular inputs [Kannan-Tetali-Vempala '99]

Negative Result for Regular Inputs

Thm: Exponential mixing time for regular instances.

Proof idea: consider this input (all required degrees are 1)

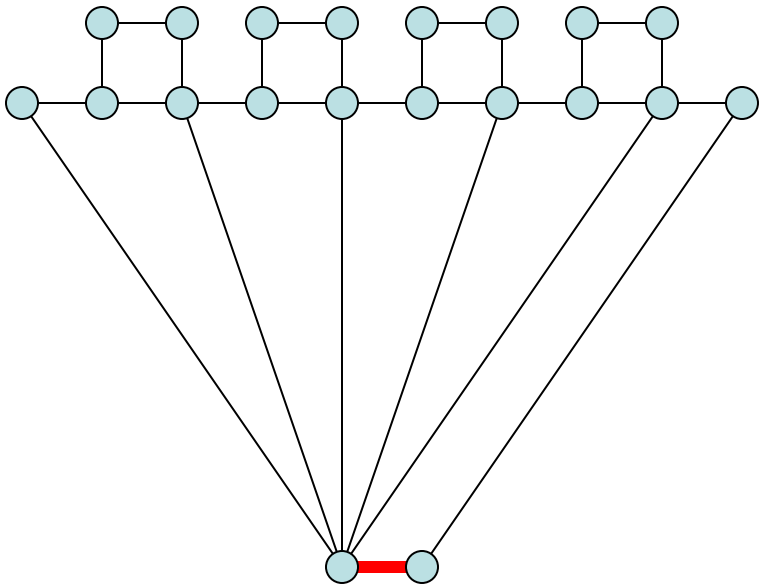


Negative Result for Regular Inputs

Thm: Exponential mixing time for regular instances.

Proof idea: consider this input (all required degrees are 1)

How many graphs contain the bottom edge ?



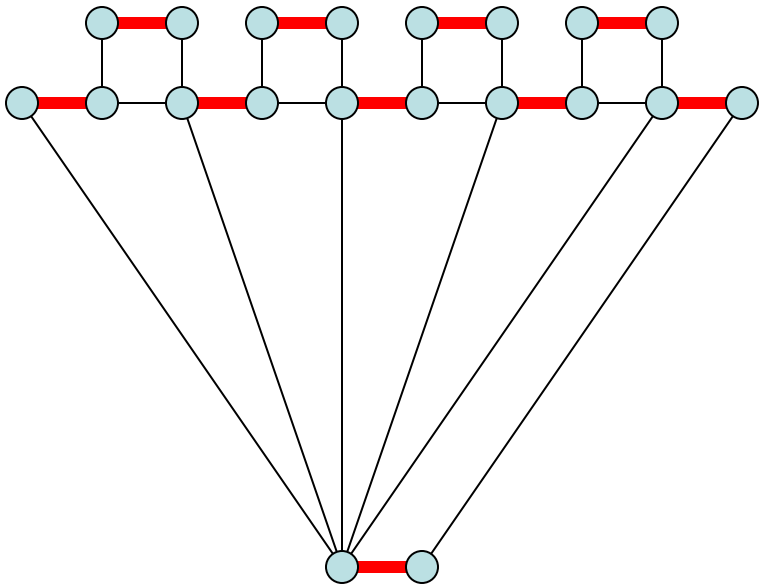
Negative Result for Regular Inputs

Thm: Exponential mixing time for regular instances.

Proof idea: consider this input (all required degrees are 1)

How many graphs contain the bottom edge ?

1



Negative Result for Regular Inputs

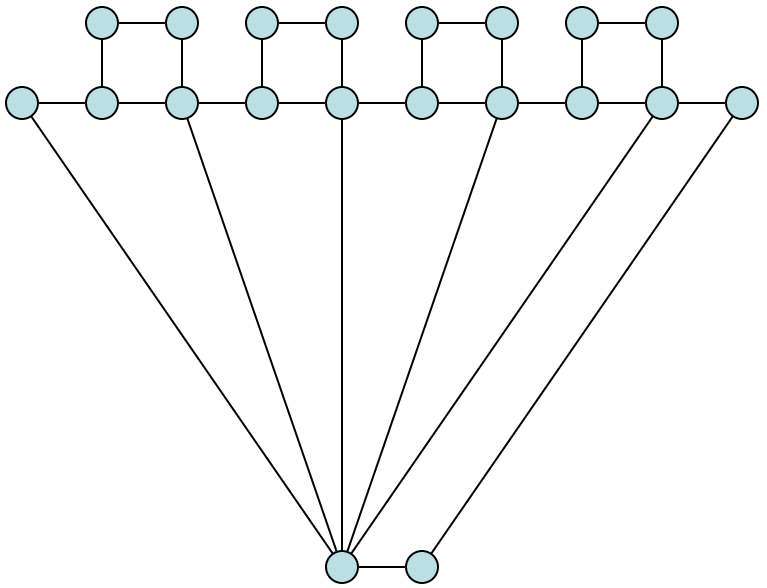
Thm: Exponential mixing time for regular instances.

Proof idea: consider this input (all required degrees are 1)

How many graphs contain the bottom edge ?

1

How many graphs overall ?



Negative Result for Regular Inputs

Thm: Exponential mixing time for regular instances.

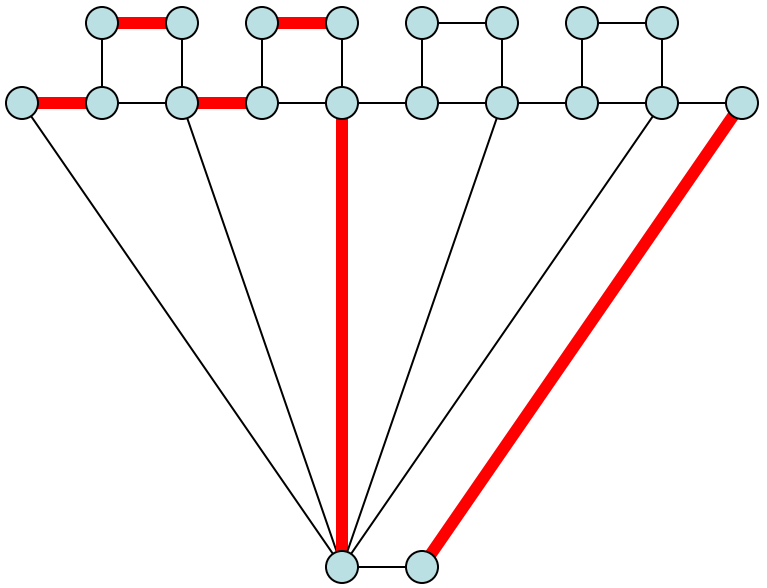
Proof idea: consider this input (all required degrees are 1)

How many graphs contain the bottom edge ?

1

How many graphs overall ?

exponential

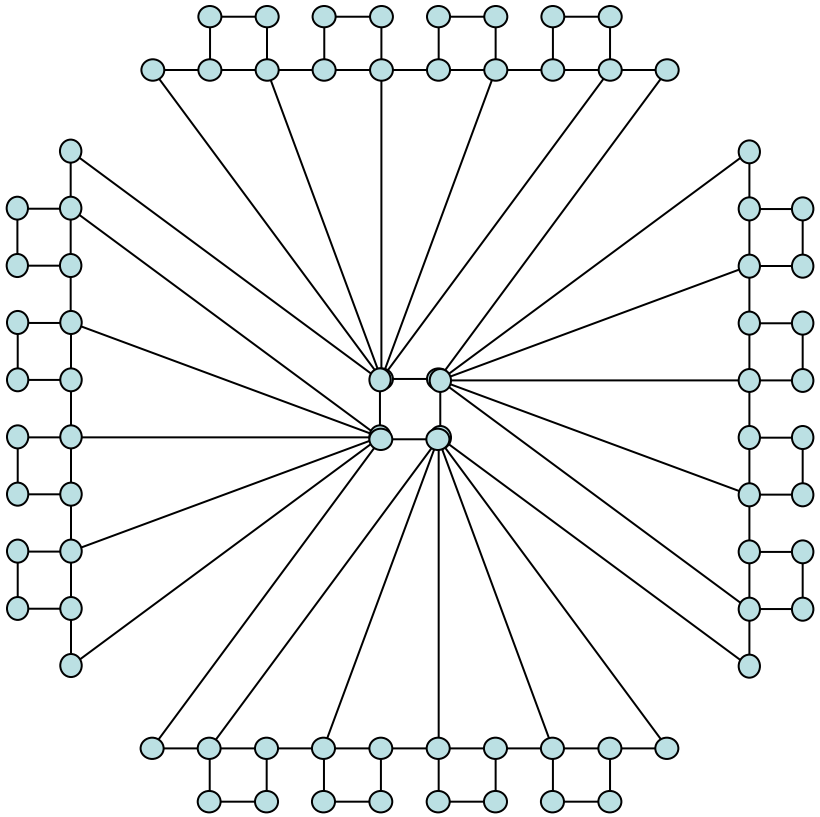


Negative Result for Regular Inputs

Thm: Exponential mixing time for regular instances.

Proof idea:

Let's connect four copies:



We have to verify:

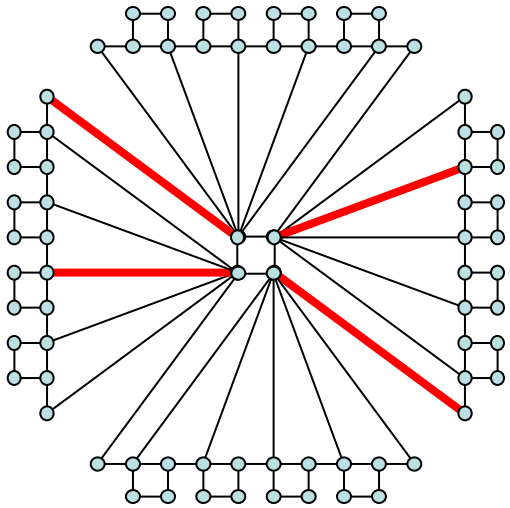
- connected state space

- exponentially small conductance

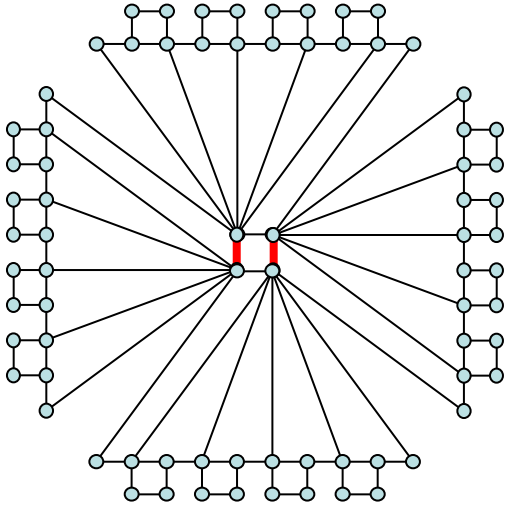
Negative Result for Regular Inputs

Thm: Exponential mixing time for regular instances.

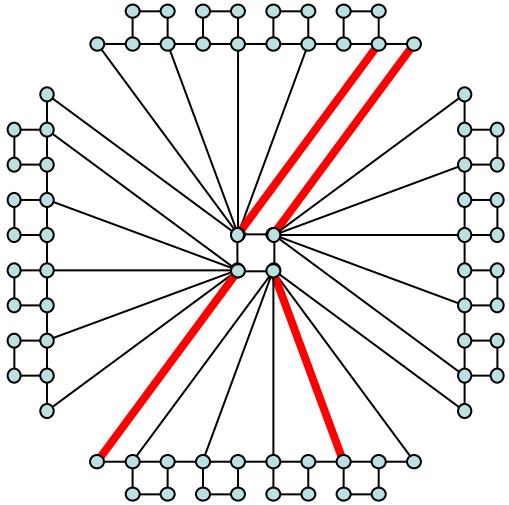
Proof idea: schematic picture of the conductance bound:



exponential



one

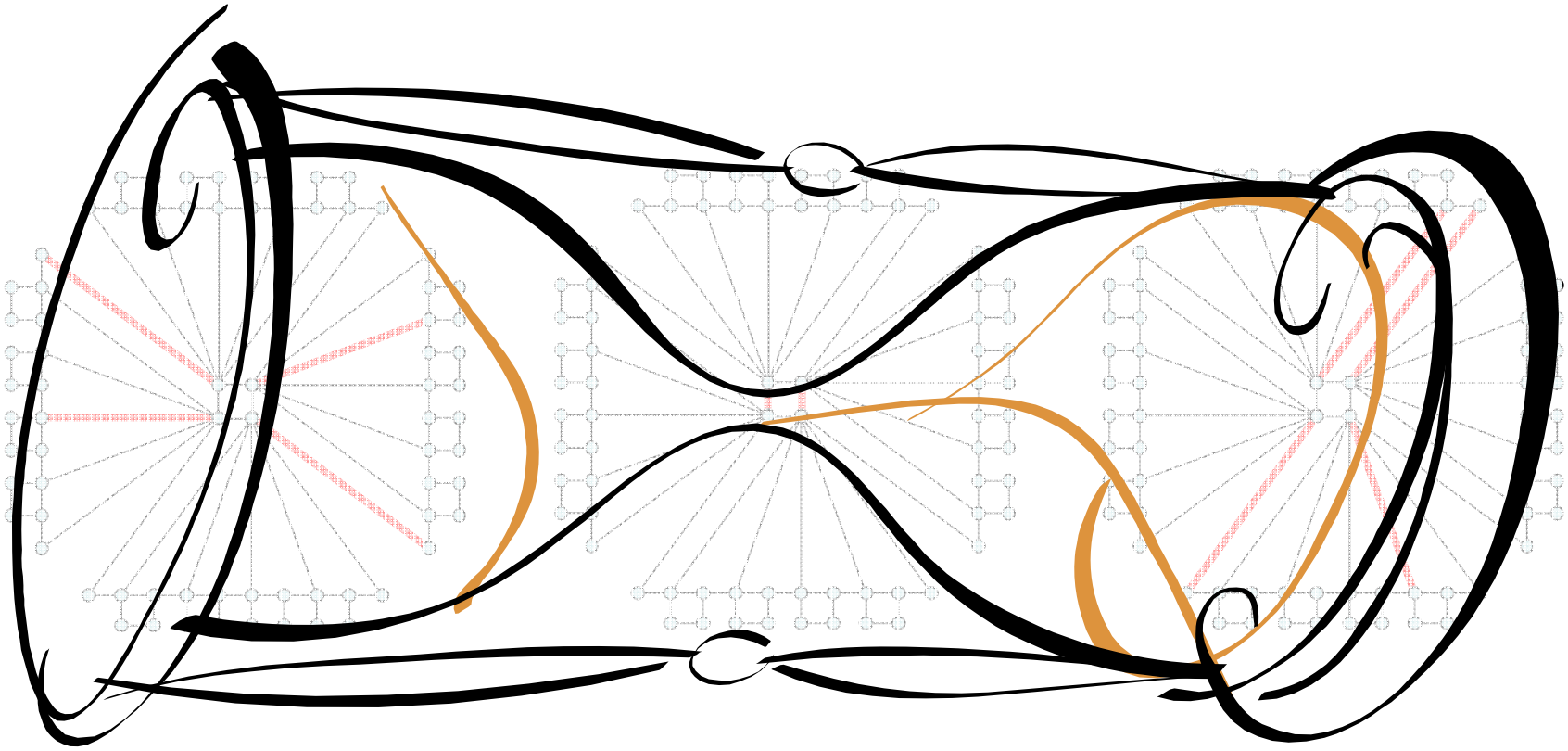


exponential

Negative Result for Regular Inputs

Thm: Exponential mixing time for regular instances.

Proof idea: schematic picture of the conductance bound:



Open Problems

- Diaconis-Gangolli MC without cell-bounds
- Diaconis-Gangolli MC with all cell-bounds 1 (non-regular inputs)
- Other approaches to sampling contingency tables without cell-bounds

- Dense regular instances ?
- All cell-bounds nonzero ?
- The presented instances are "just-connected," are there slowly mixing instances that would be more connected ?