Counting minimum (s,t)-cuts in weighted planar graphs in polynomial time

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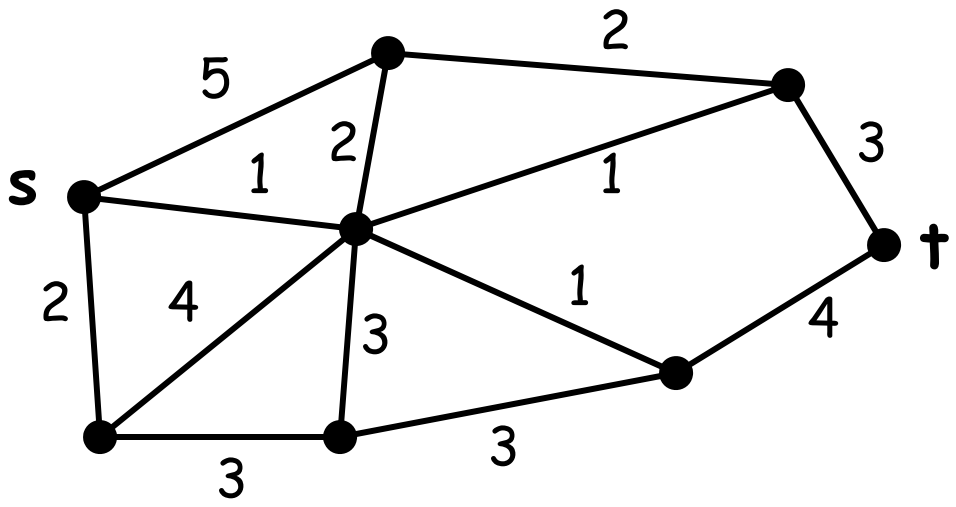
Theory Canal, September 20, 2010

The problem: Counting minimum (s,t)-cuts

Input: a positively weighted (directed) planar graph $G=(V,E)$ and two vertices s,t

Output: the number of minimum (s,t)-cuts of G

Example:



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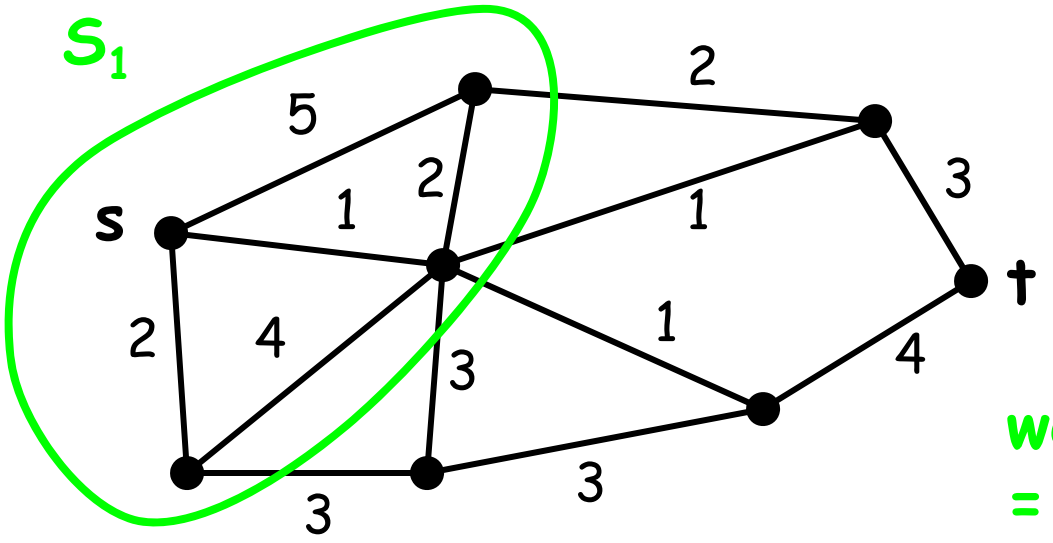

$weight(S_1)$
$= 2+1+1+3+3$
$= 10$

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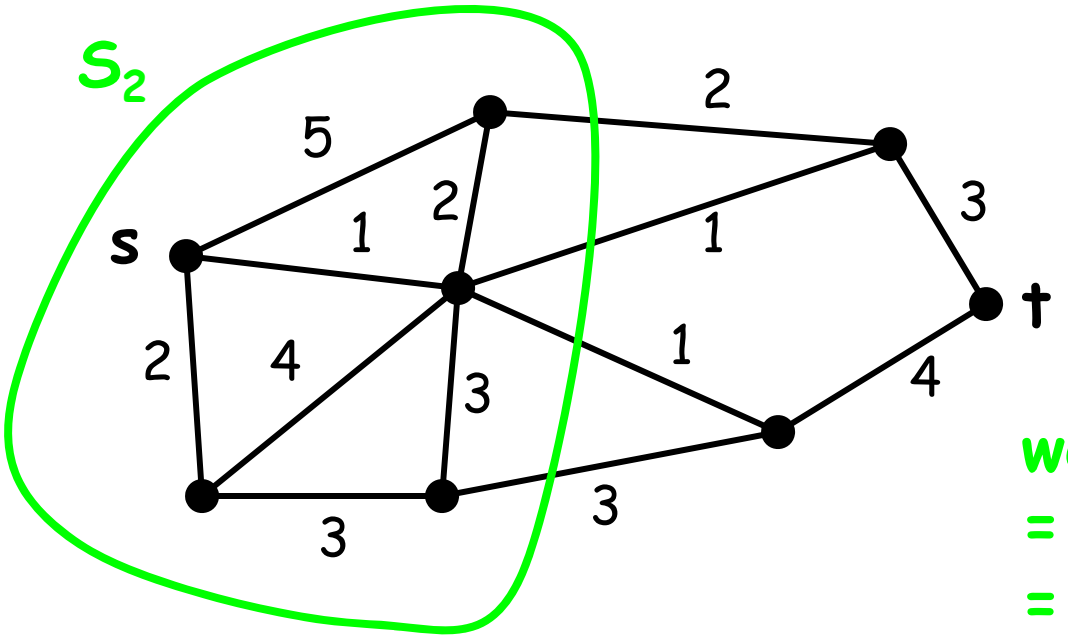

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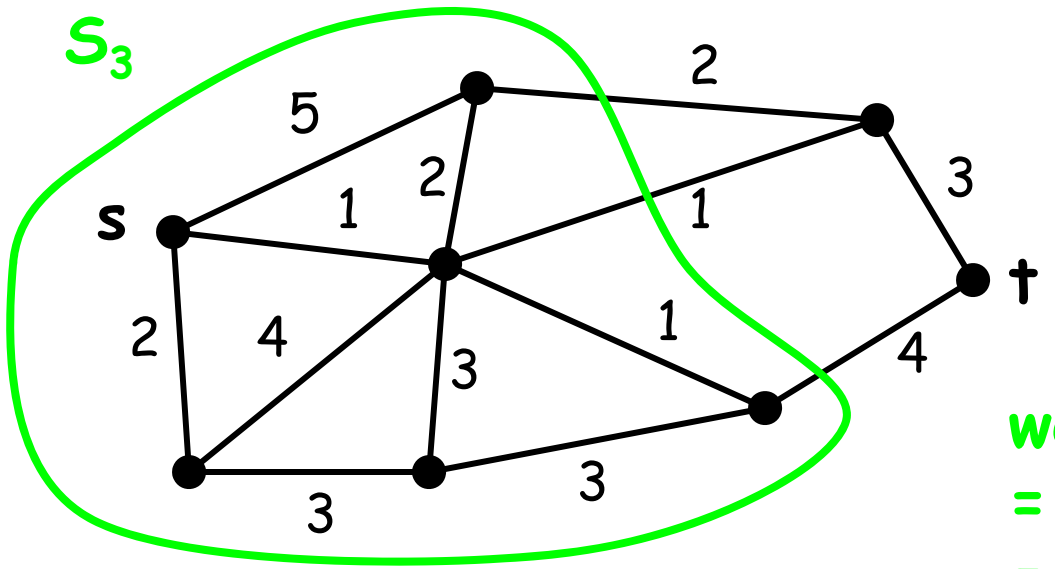

weight(S_3)
= 2+1+4
= 7

Recall: (s,t)-cut is a set $S \subseteq V$ containing s but not t; its weight is the sum of edge weights out of S

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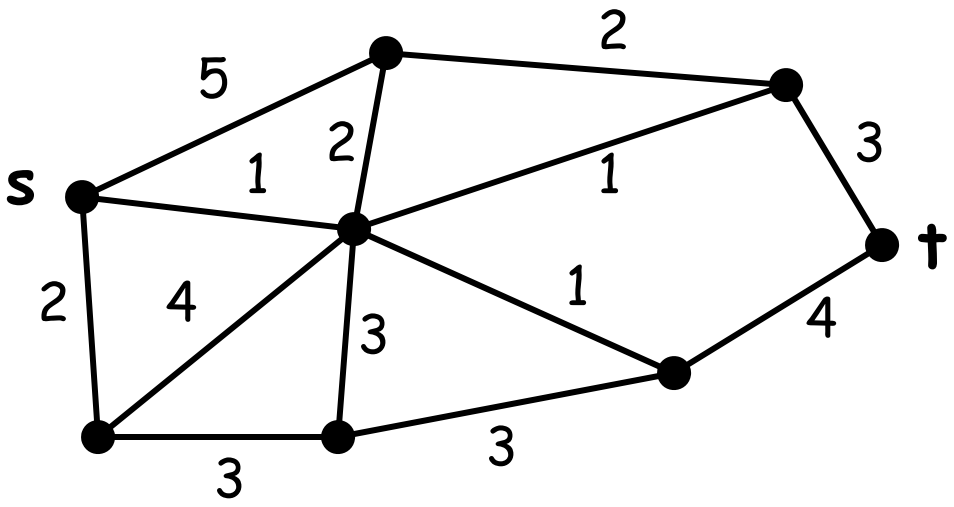

Min (s,t)-cut weight: 7

Number of min (s,t)-cuts: 5

Motivation & related work

Initial motivation: **network reliability problems**

- number of min (s,t)-cuts useful in estimating the probability of disconnecting the network, e.g., [Colbourn '05]

- most related: Ball and Provan '82-'84

- [Ball and Provan '83]: efficient poly-time counting in (unweighted) planar (multi-) graphs when s,t on the same face

- [Nagamochi, Sun, Ibaraki '91]: unweighted multi-graphs, running time dependent on the number of cuts – not poly-time (improving Ball & Provan's run.time for non-planar graphs)

Other, recent, motivation: **computer vision**

- influential works, e.g.: [Boykov, Veksler & Zabih '01], [Boykov & Kolmogorov '04], [Geman & Geman '84]

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- the simplest case: image segmentation where image represented by a planar graph

- user selects two points, the graph cut represents the segmentation

- currently in use only min-cut algorithms (optimization version), using an arbitrary min-cut

http://path.upmc.edu/cases/case123.html

- many advantages of counting (and the related sampling) versions, e.g.:

  - user can choose from several cuts

  - can be used to compute the partition function that can be used to estimate model parameters

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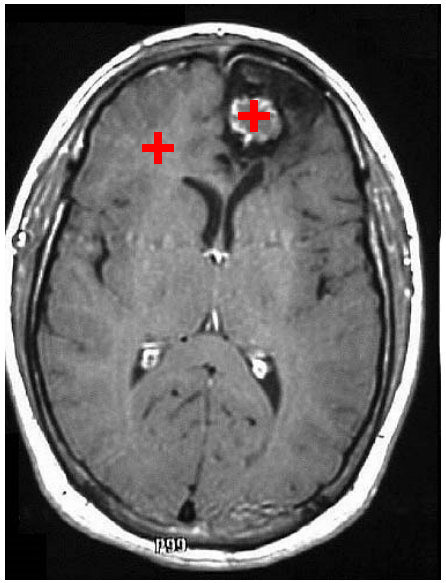

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Our contributions

Thm: An $O(nd + n \log n)$ algorithm computing the number of minimum (s,t)-cuts in weighted planar graphs, where:

- n = # vertices

- d = length of the shortest s-t path

- for directed graphs, assume all vertices are reachable from s and lead to t.

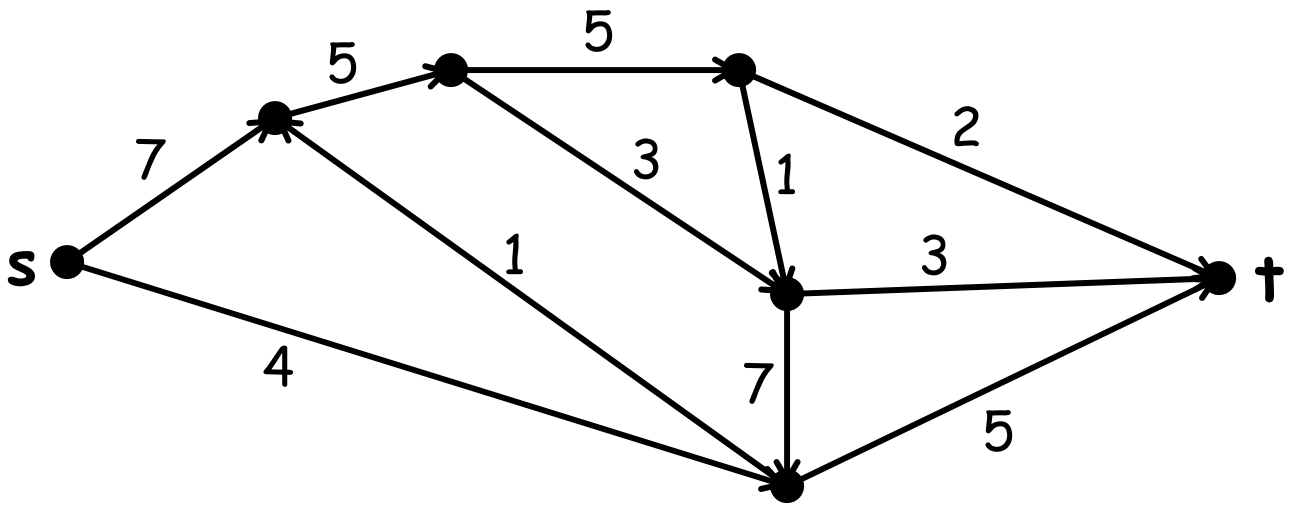
Comparison with earlier works:
here no assumptions on the relative position of s and t (and weighted)

-> important for computer vision applications

Review of network flows

Flow network:

- a directed graph with positive capacities on the edges, and
- two vertices s (the source) and t (the sink)



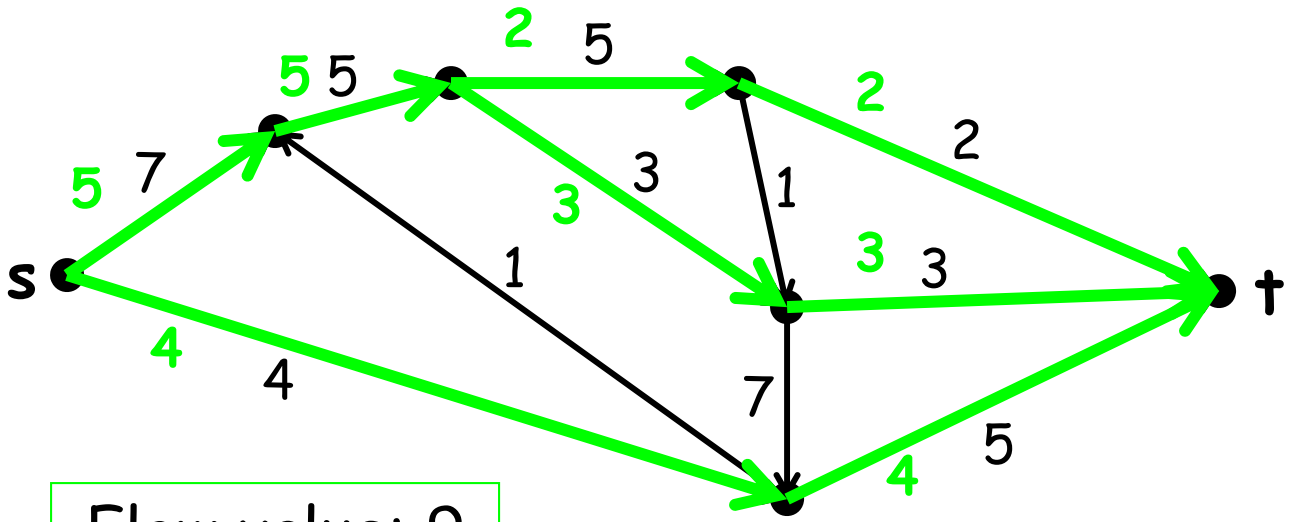
Review of network flows

Flow f: flow amount on every edge satisfying:

- for every edge e: flow amount f(e) $\leq$ capacity c(e), and
- for every vertex v (except s,t):

flow amount into v = flow amount out of v

- flow value: amount out of s minus amount into s

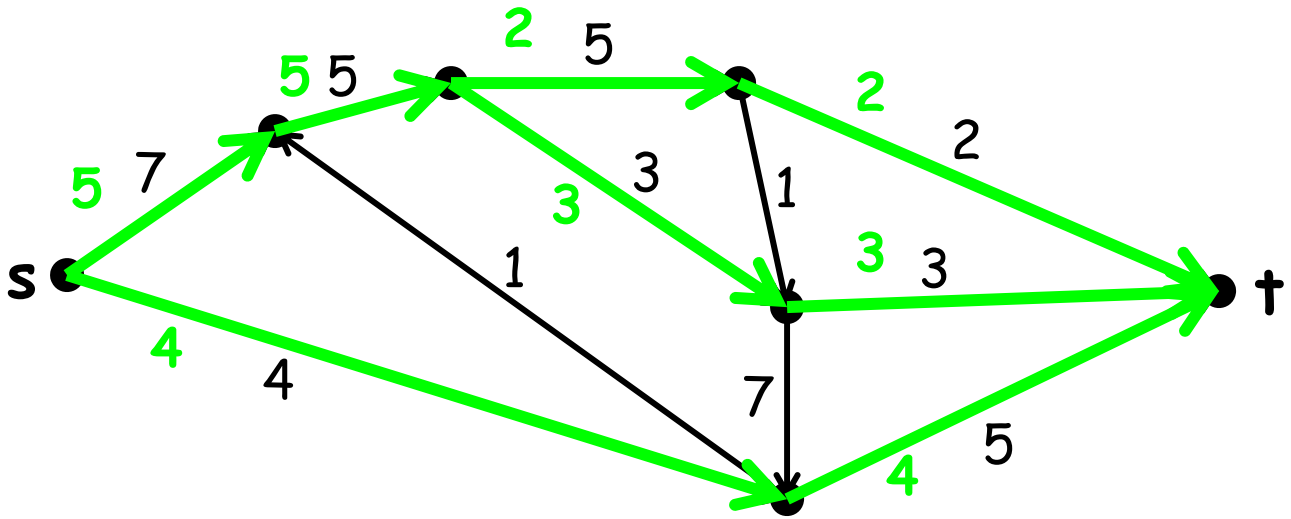


Flow value: 9

Review of network flows

Residual graph of a flow f:

- **forward edges**: weight = capacity – flow
- backward edges: weight = flow



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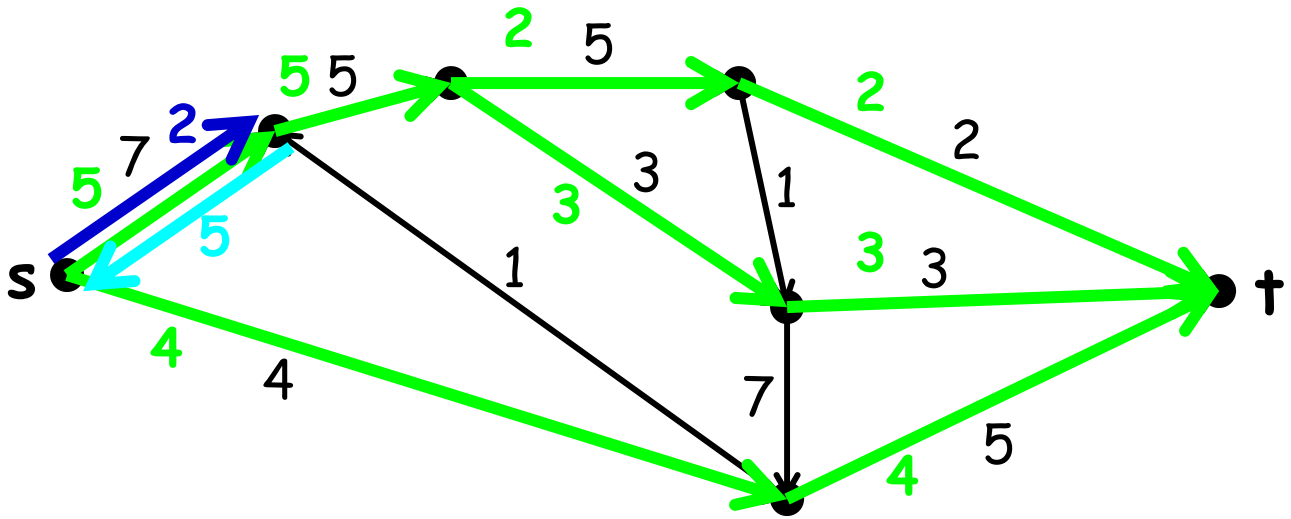

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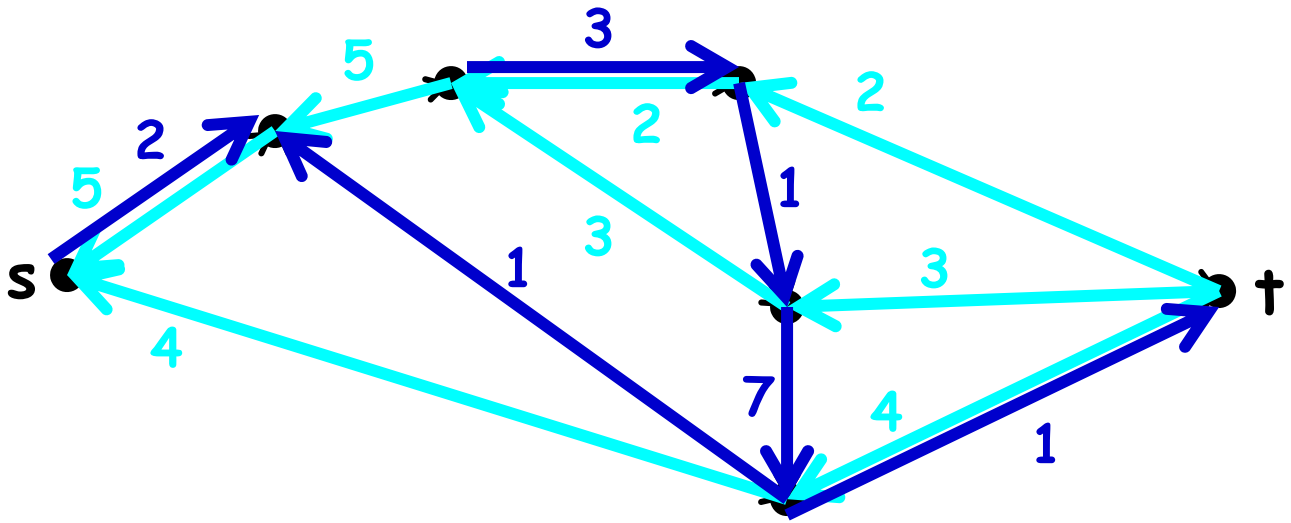
(only edges with positive weight)

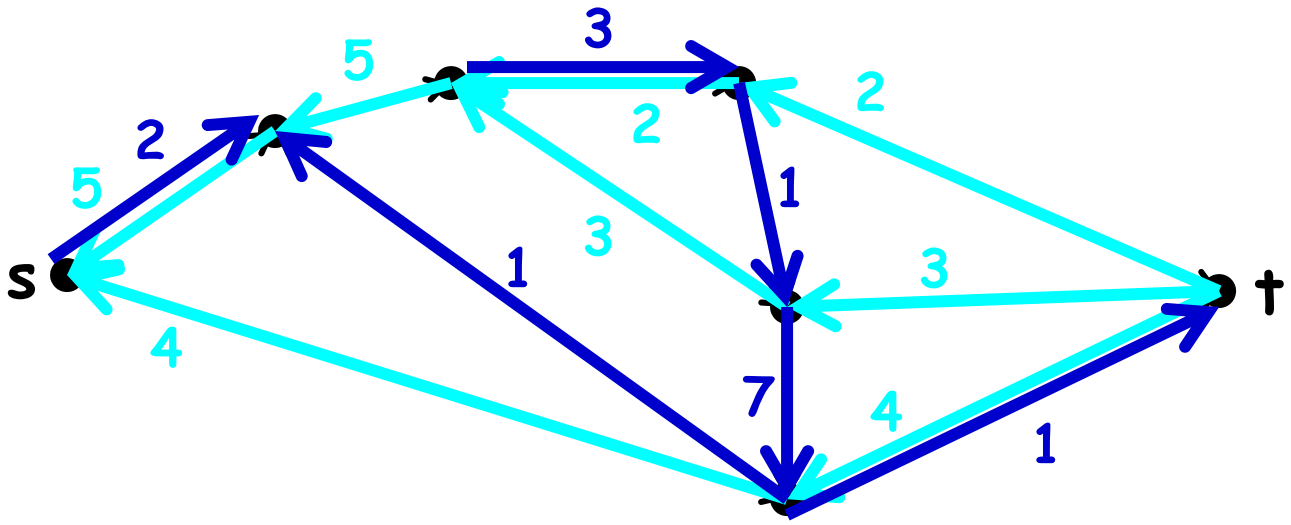


Review of network flows

Ford-Fulkerson Thm:

value of max s-t flow = value of min s-t cut

Note: flow is max iff no s-t path in the residual graph



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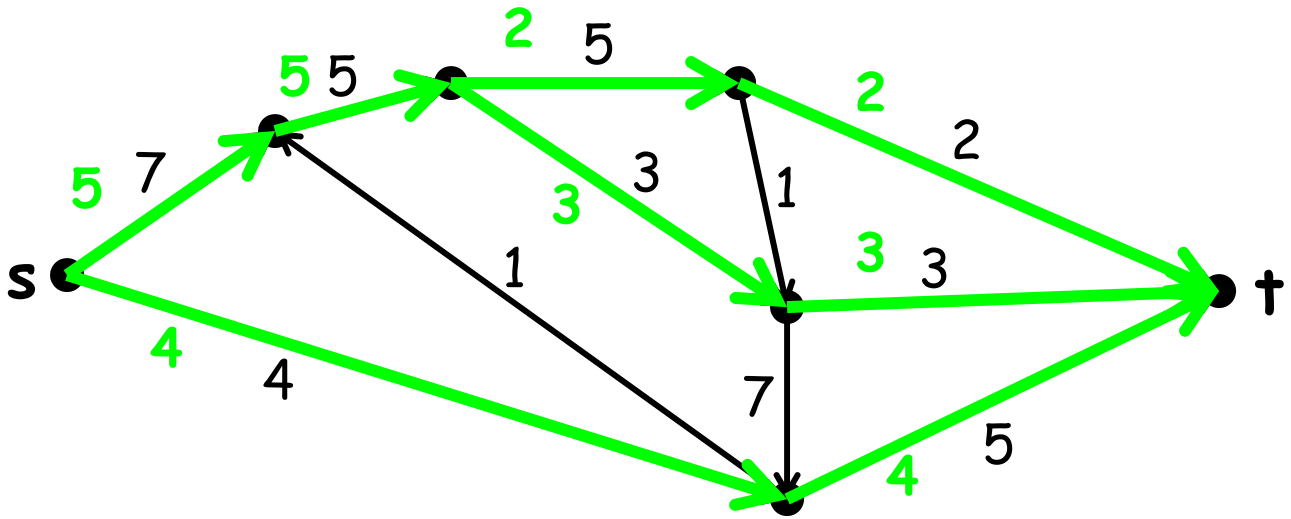

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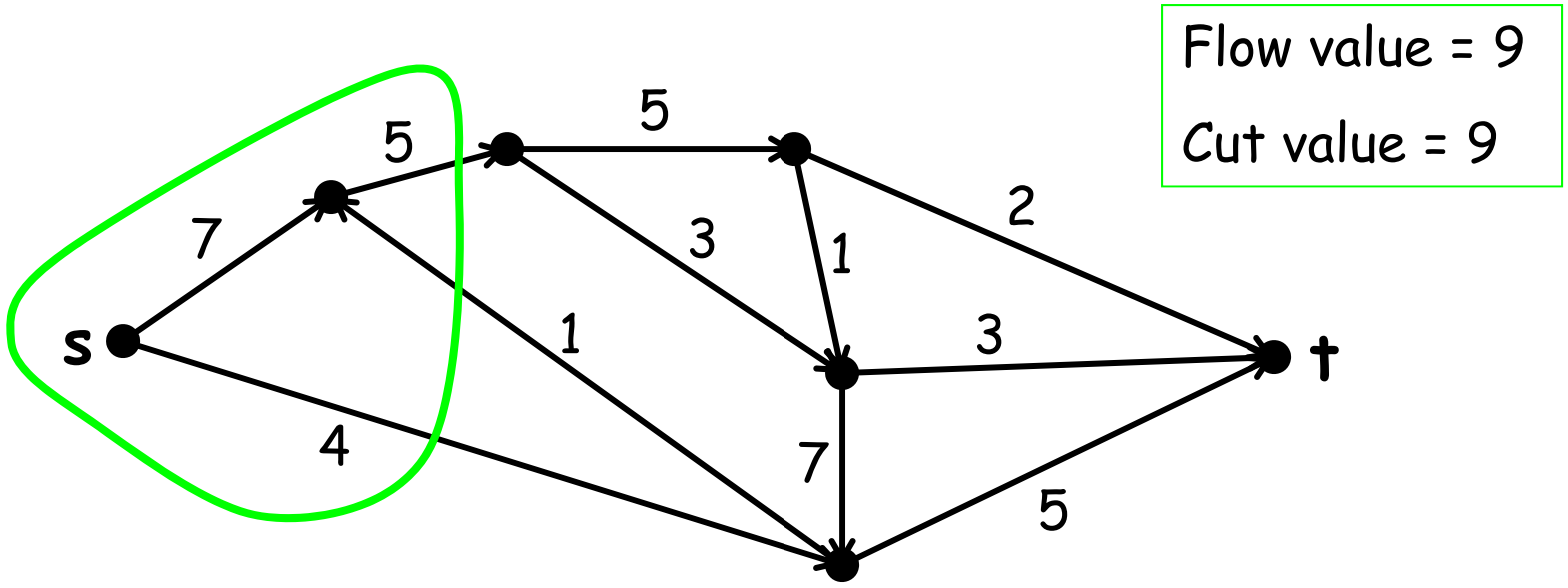
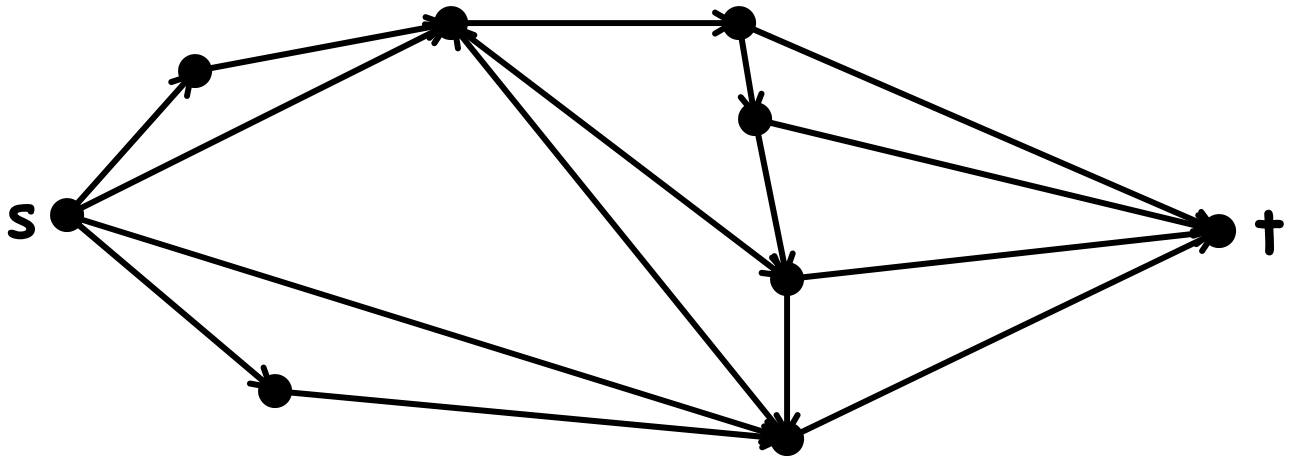


Ball & Provan's counting approach

Given an unweighted (multi-)graph:

1. Find a max flow and construct the residual graph
2. Contract strongly connected components
3. Compute # "forward-cuts" in the DAG

   (forward-cuts = maximal antichains in the poset)



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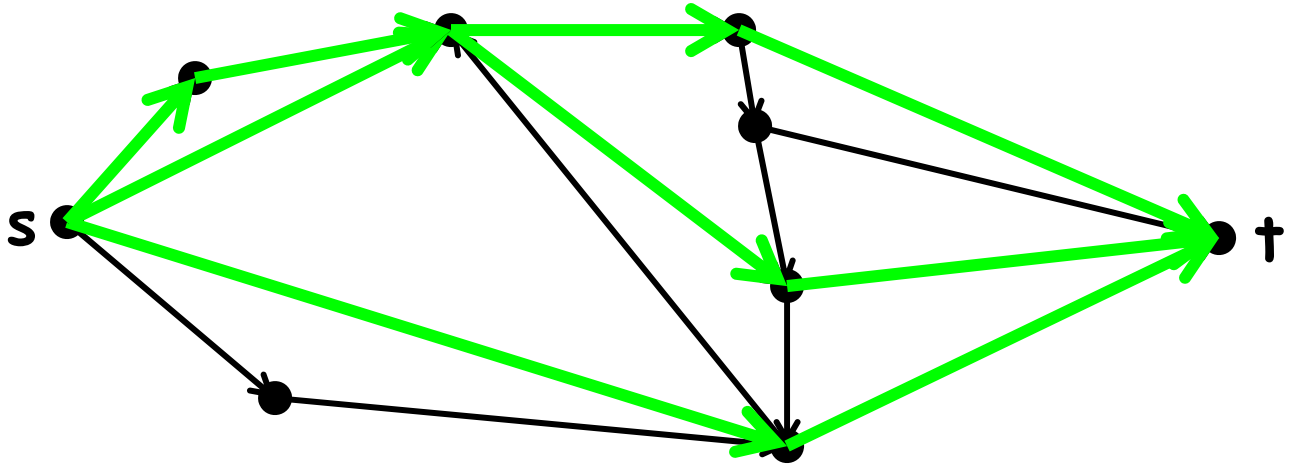

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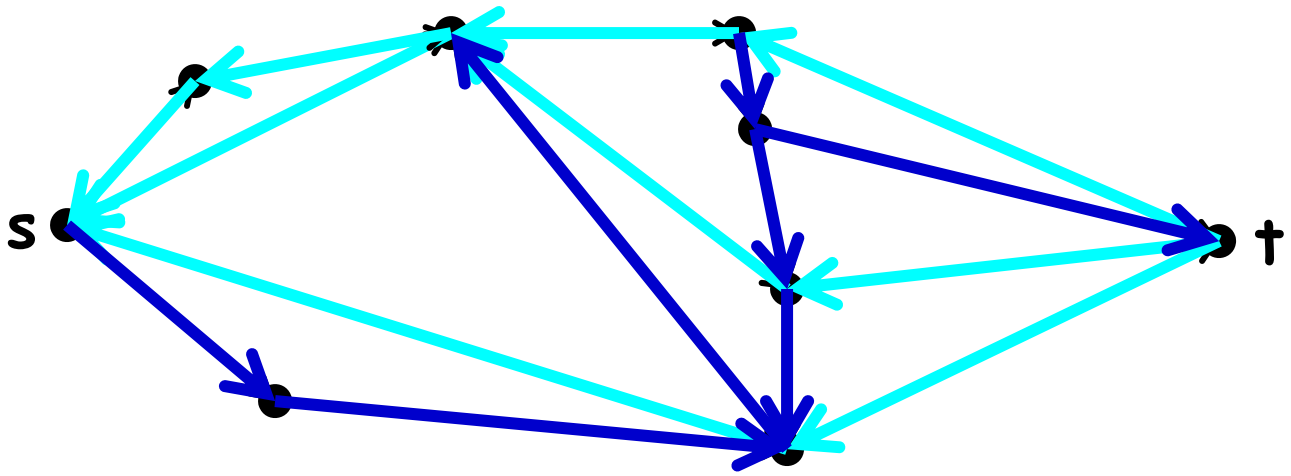

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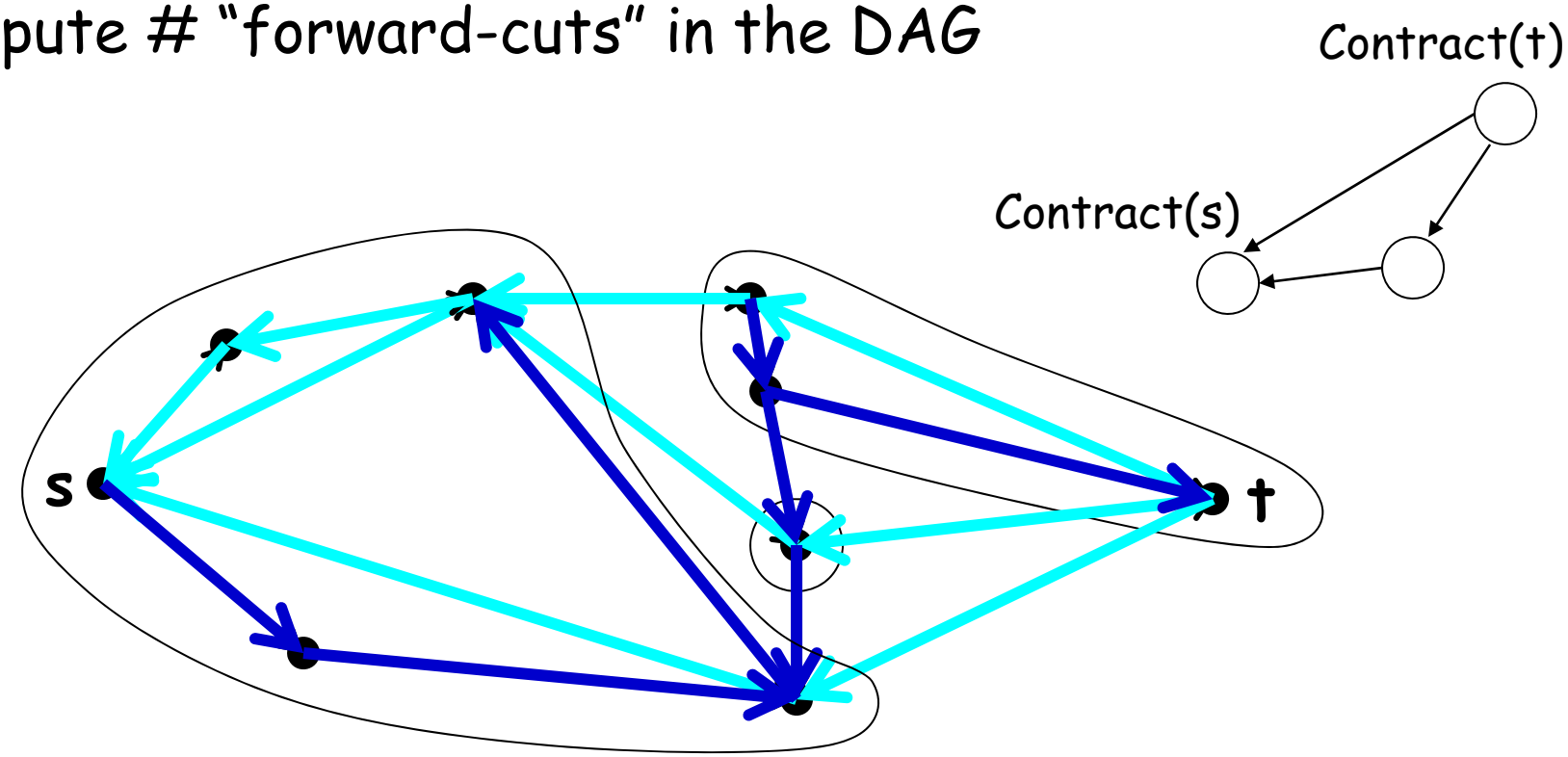

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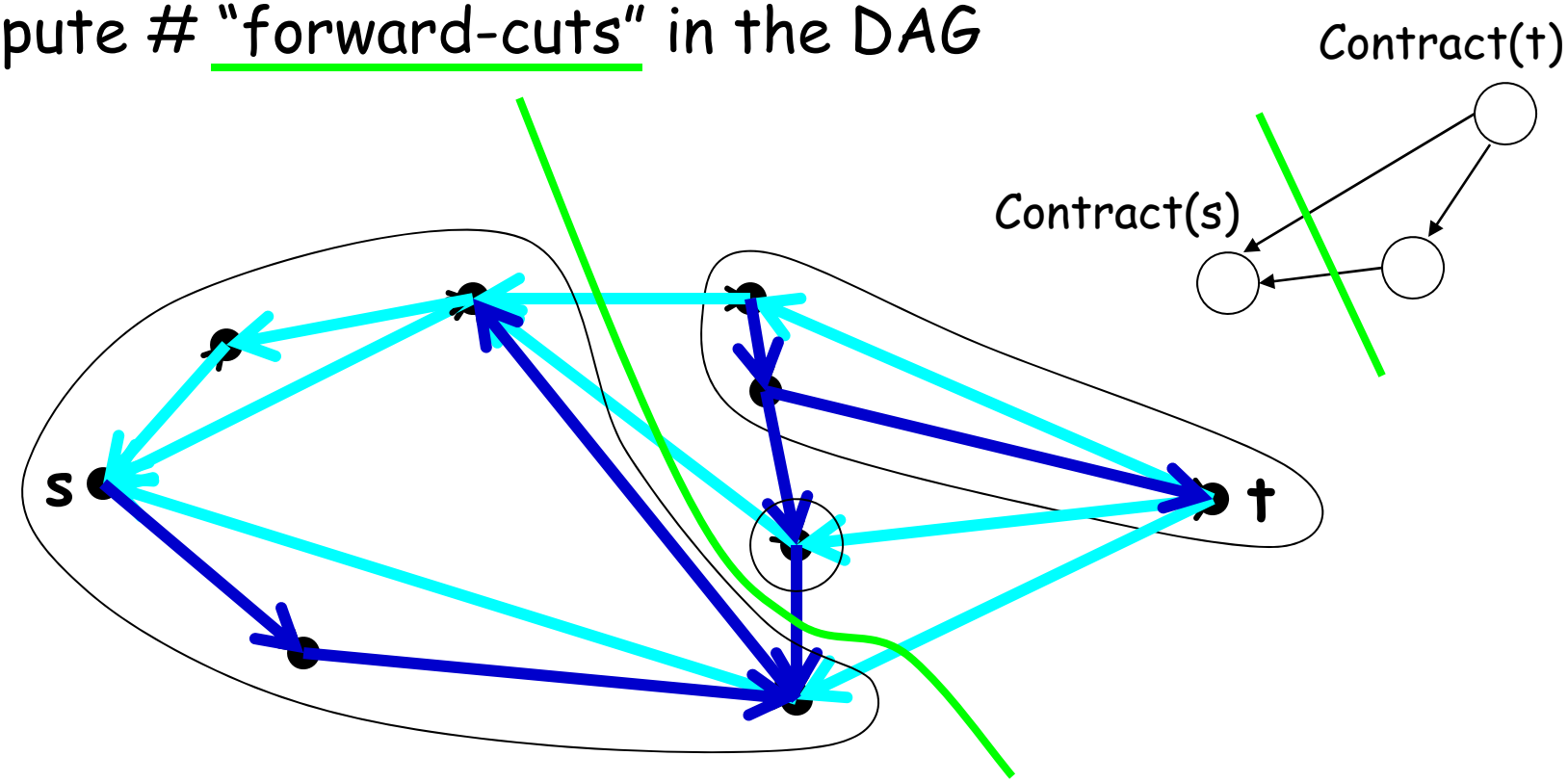

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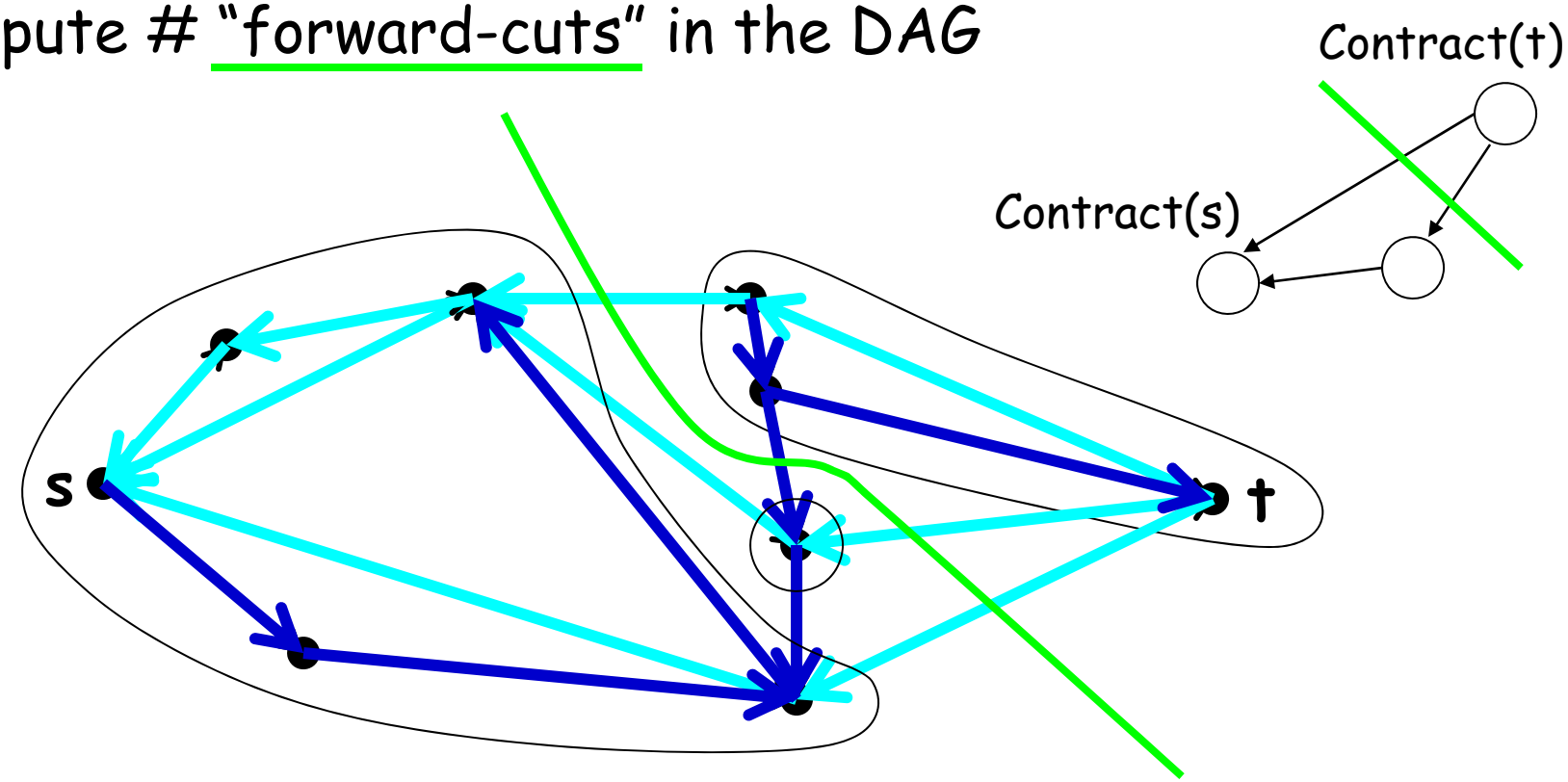

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- contains Contract(s) and not Contract(t), and
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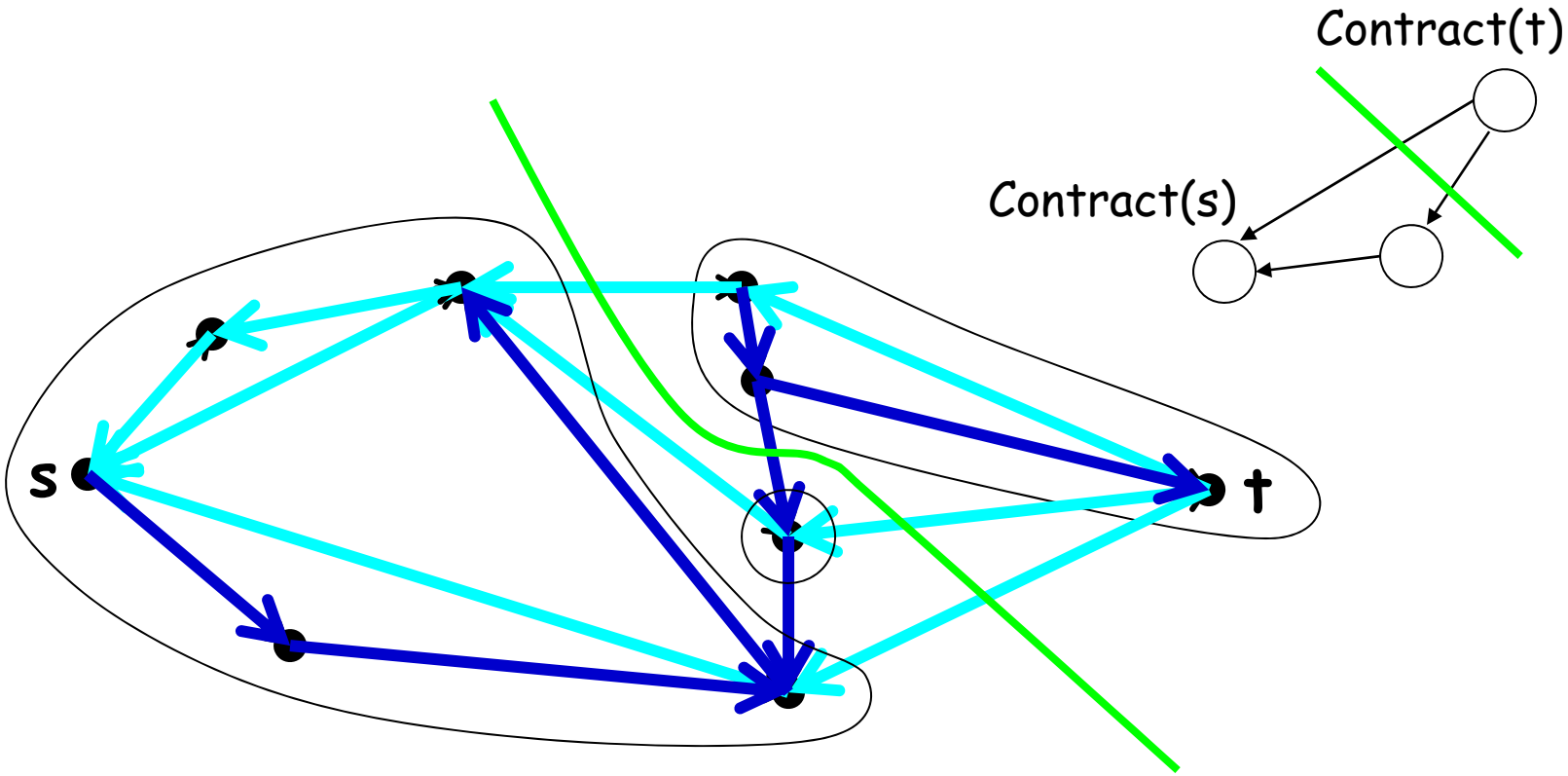

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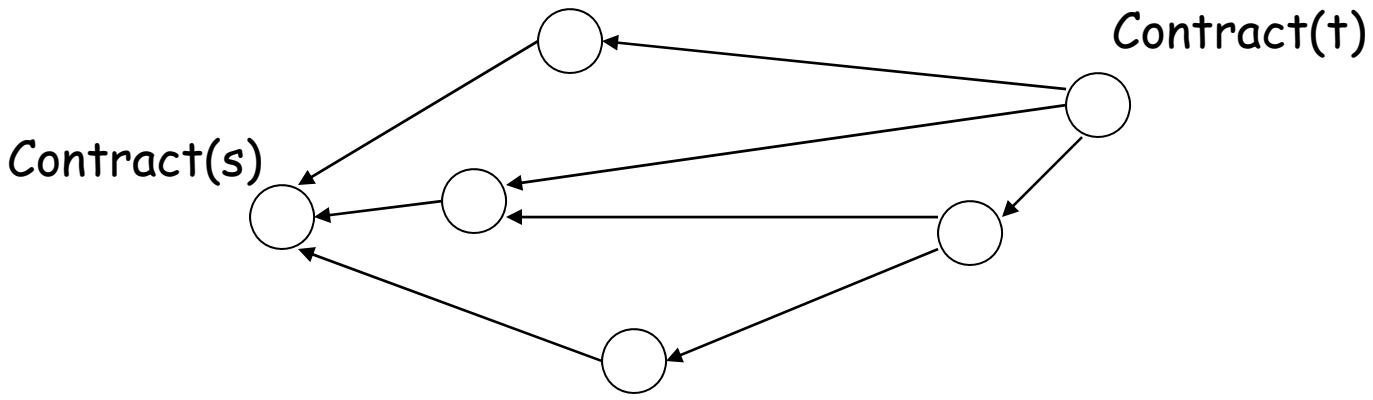

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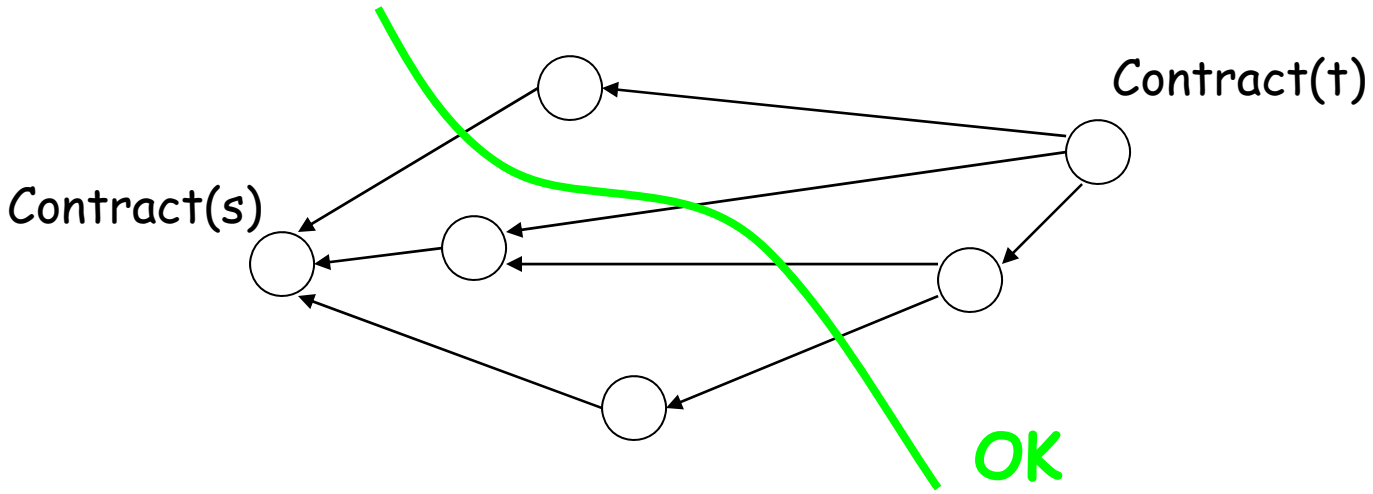

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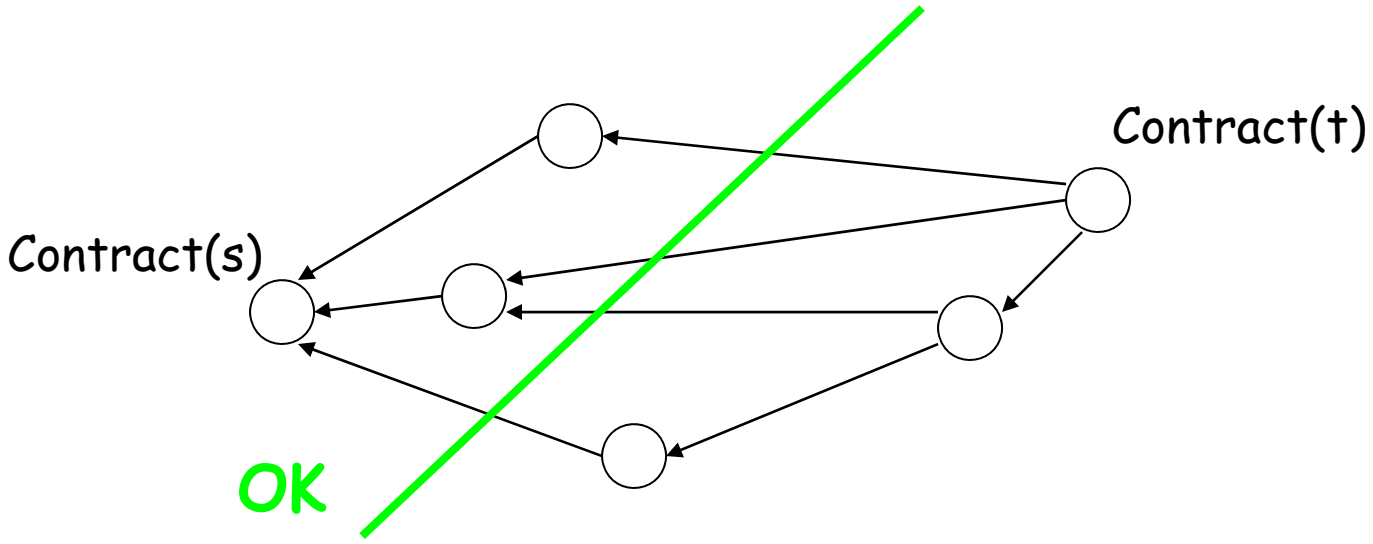

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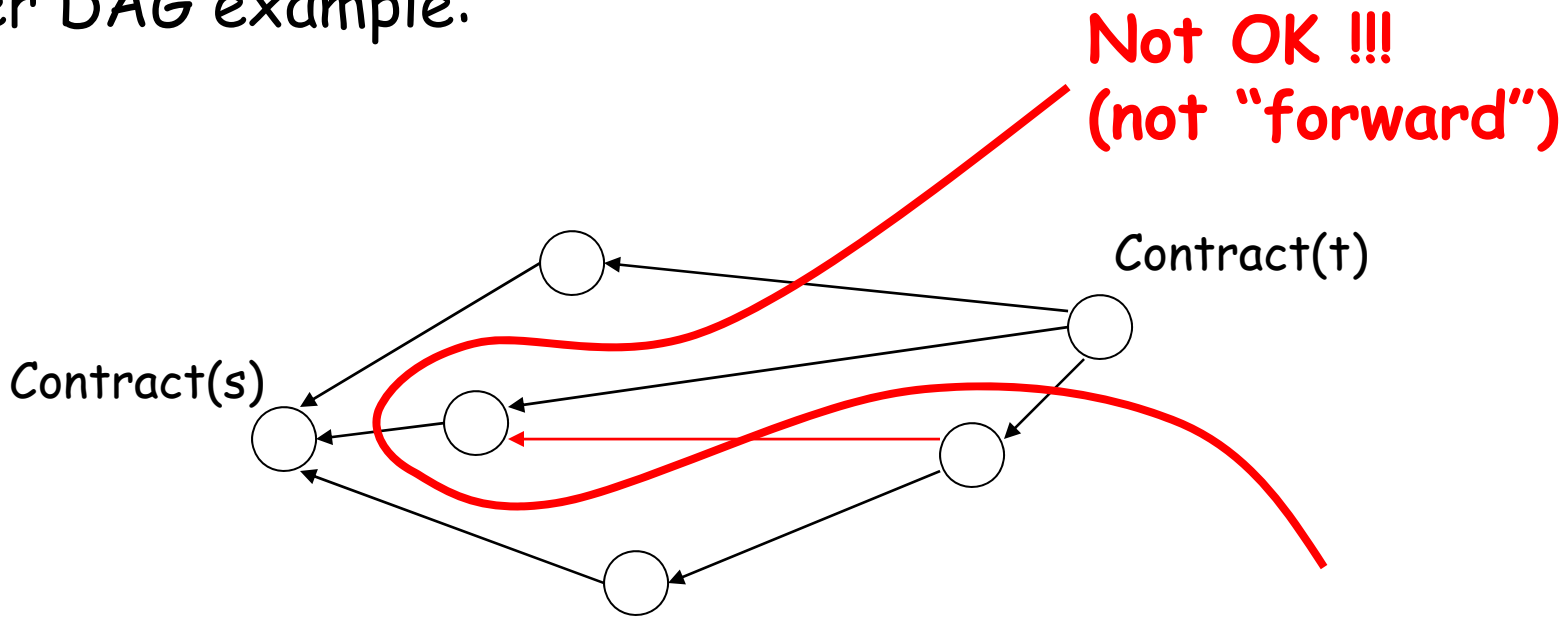
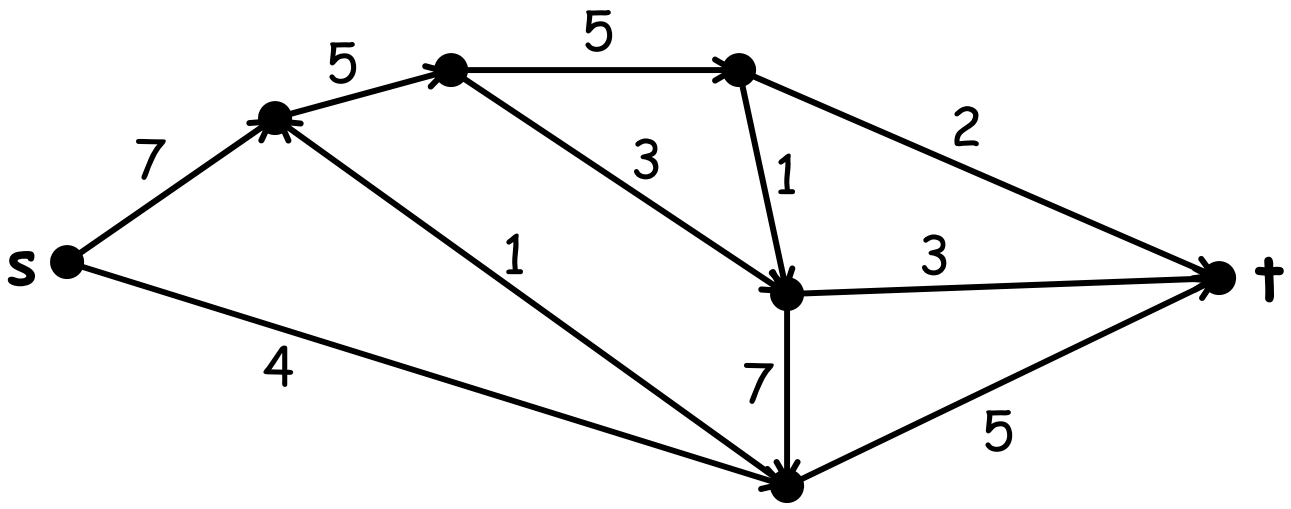


The weighted case

What if the graph is (positively) weighted ?

- the reduction to forward-cuts still works [this paper] (note: the resulting DAG is unweighted)
- also note: the reduction works for all graphs, not just planar graphs



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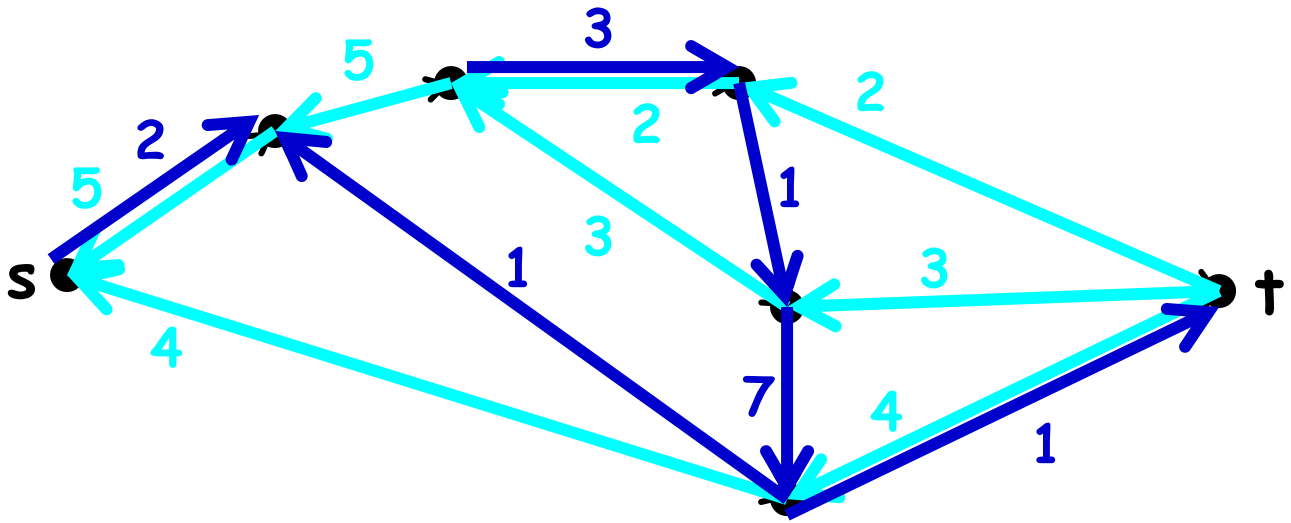

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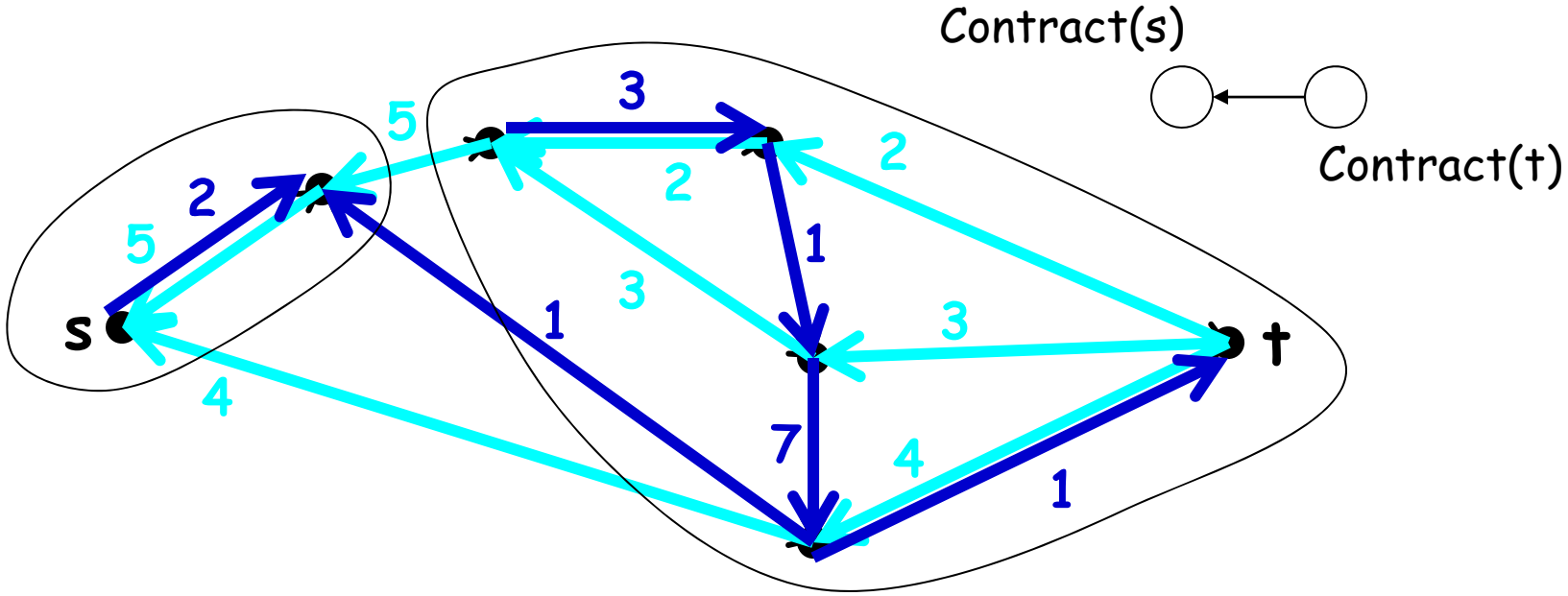

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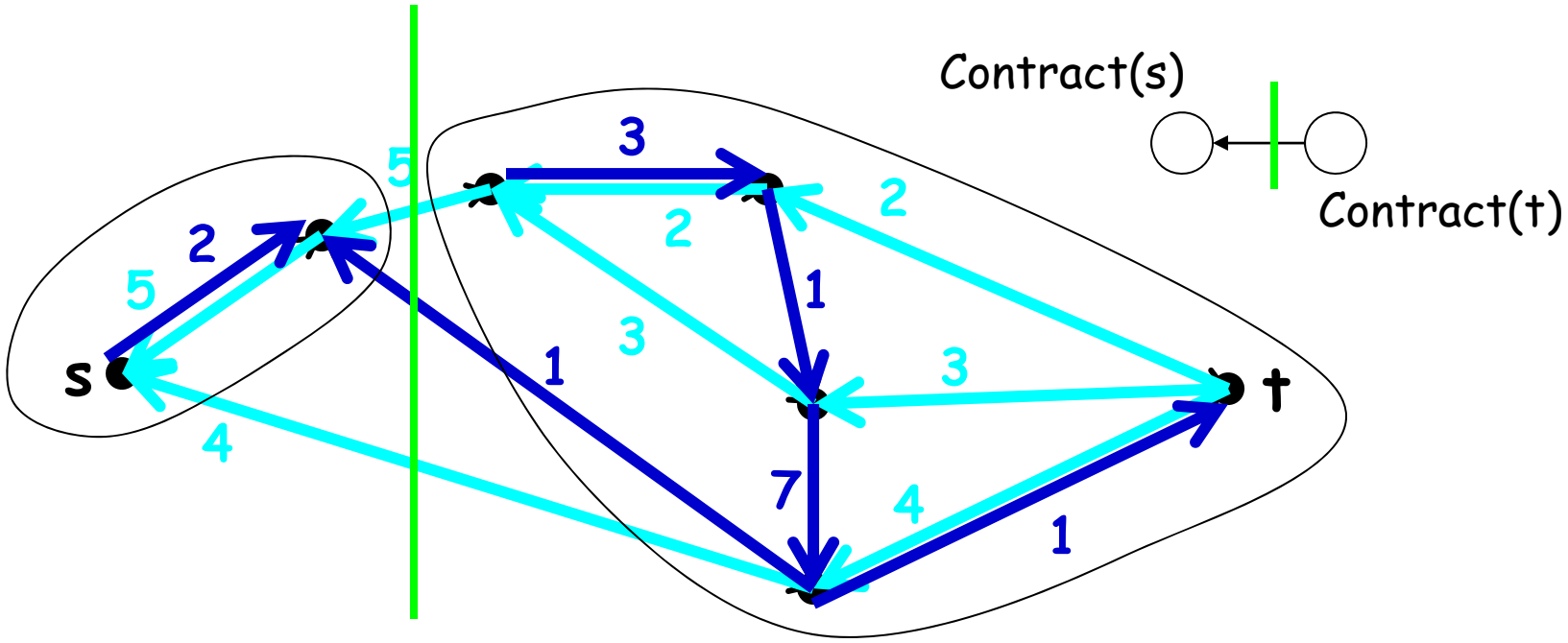
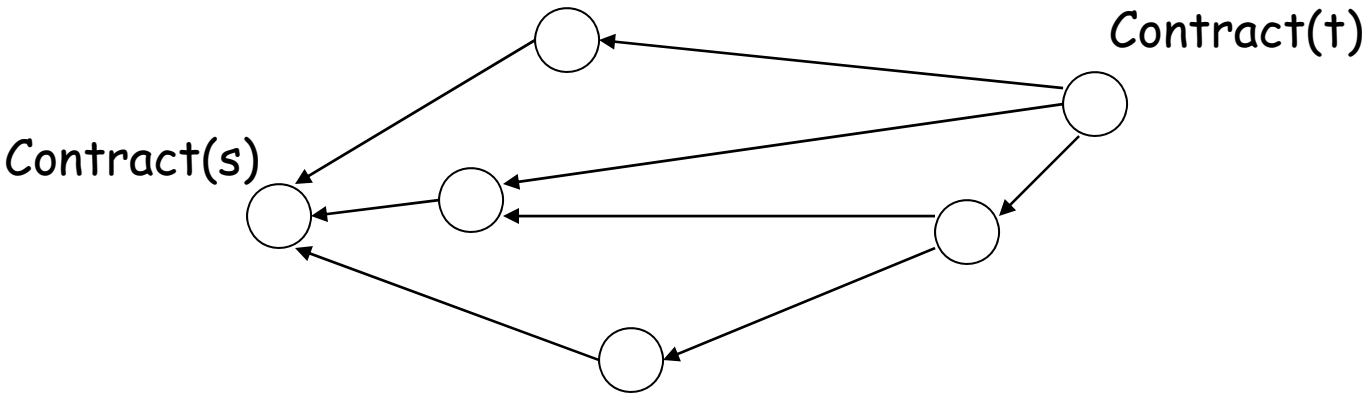


"Forward-cuts" in planar DAGs

Observe: Planar input graph -> planar DAG

Goal: count "forward-cuts" (or maximal antichains)



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- split the outer face into the "top" and the "bottom" face
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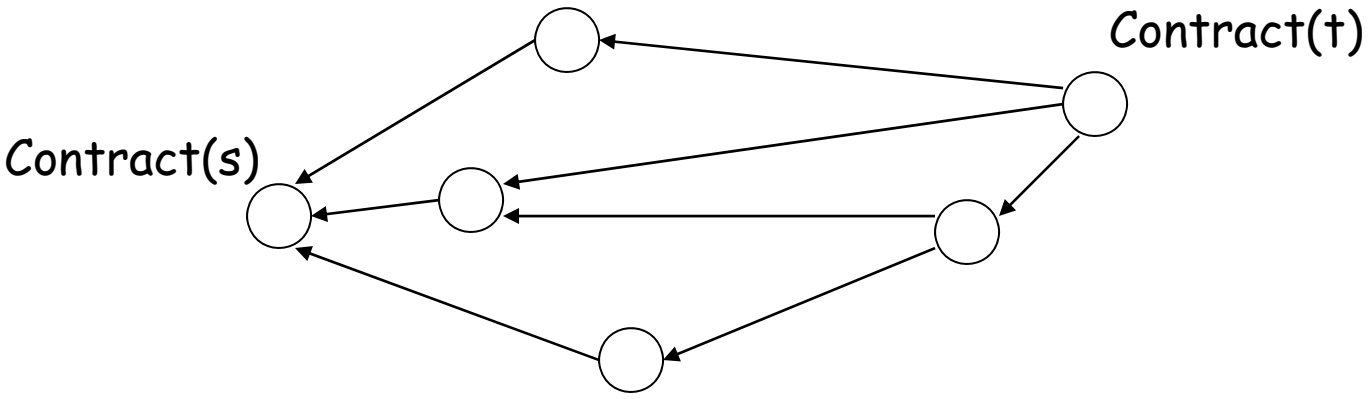

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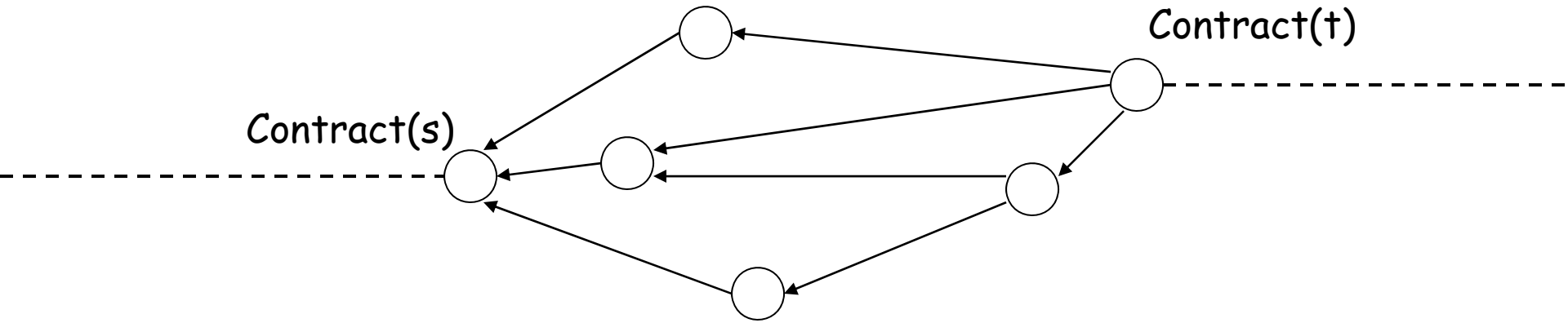

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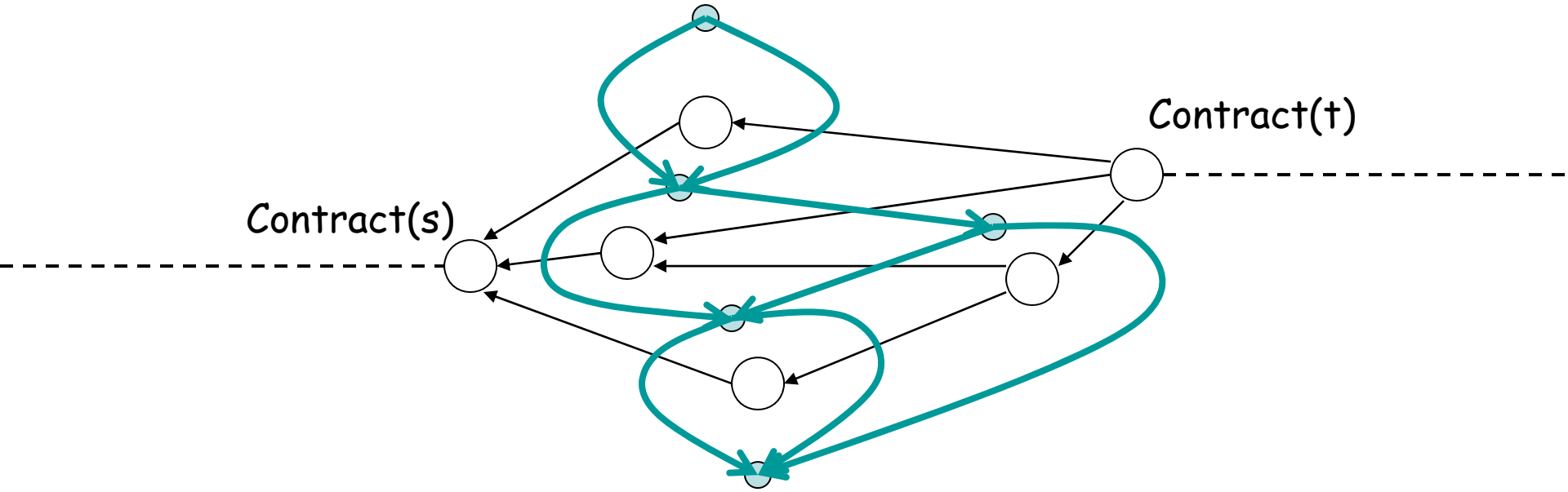

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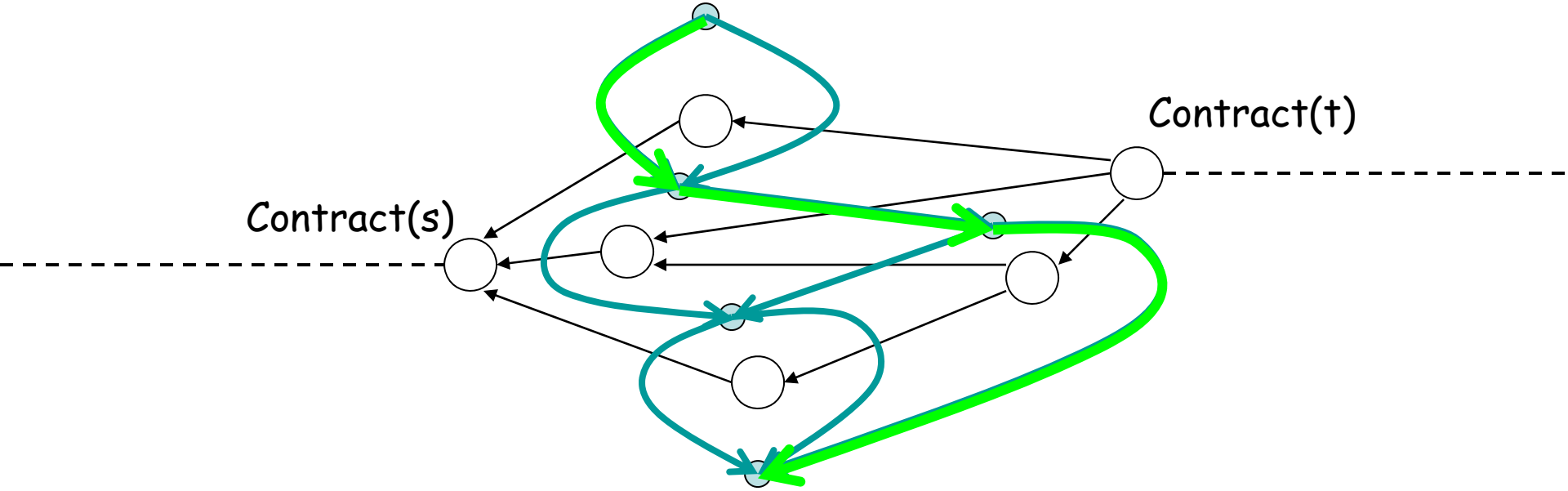

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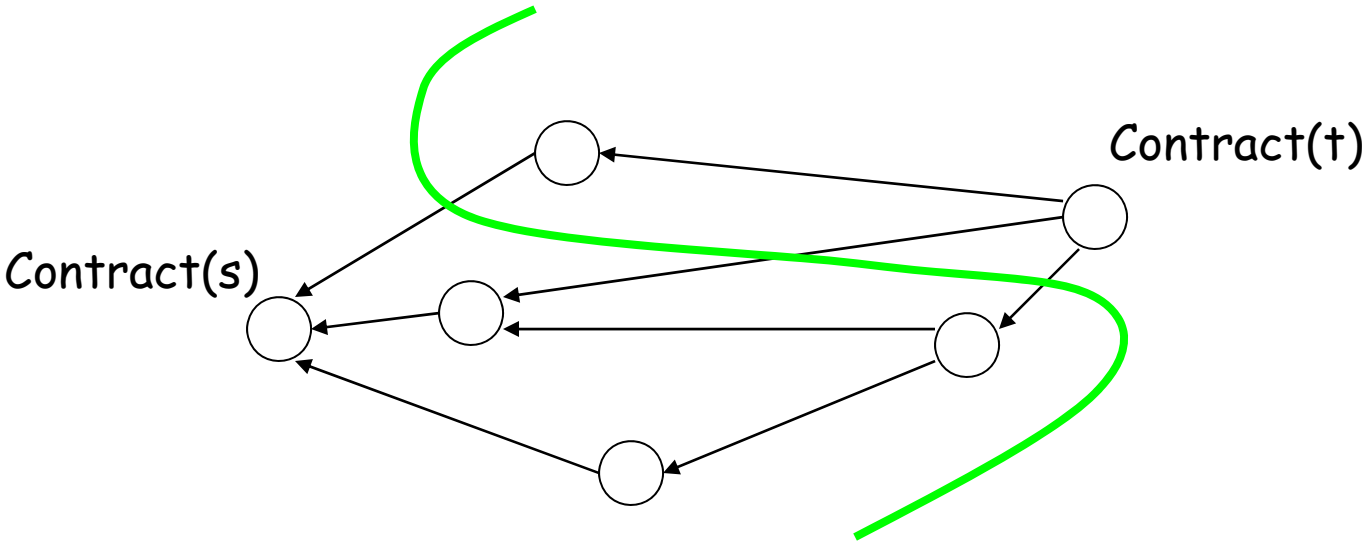

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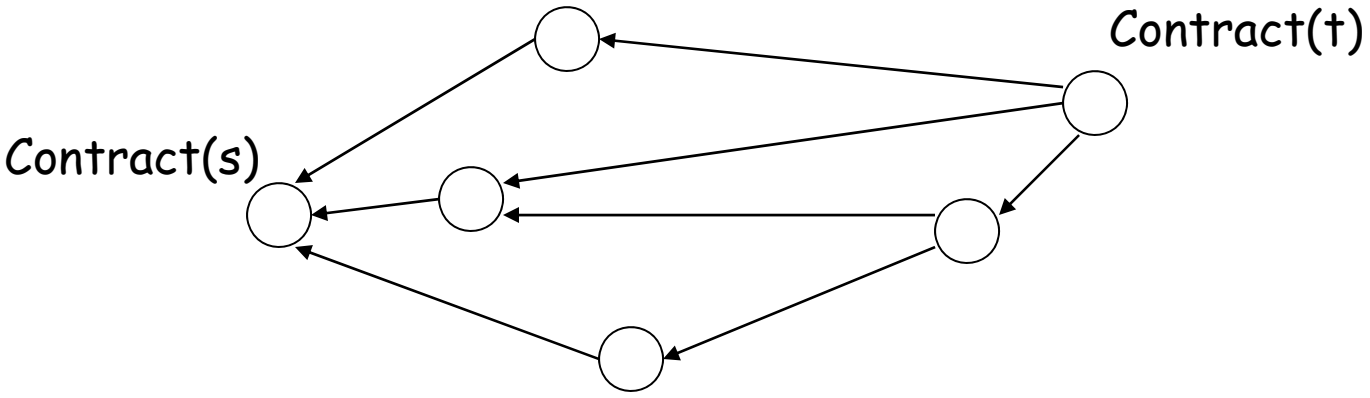
Note: poly-time because the dual is a DAG



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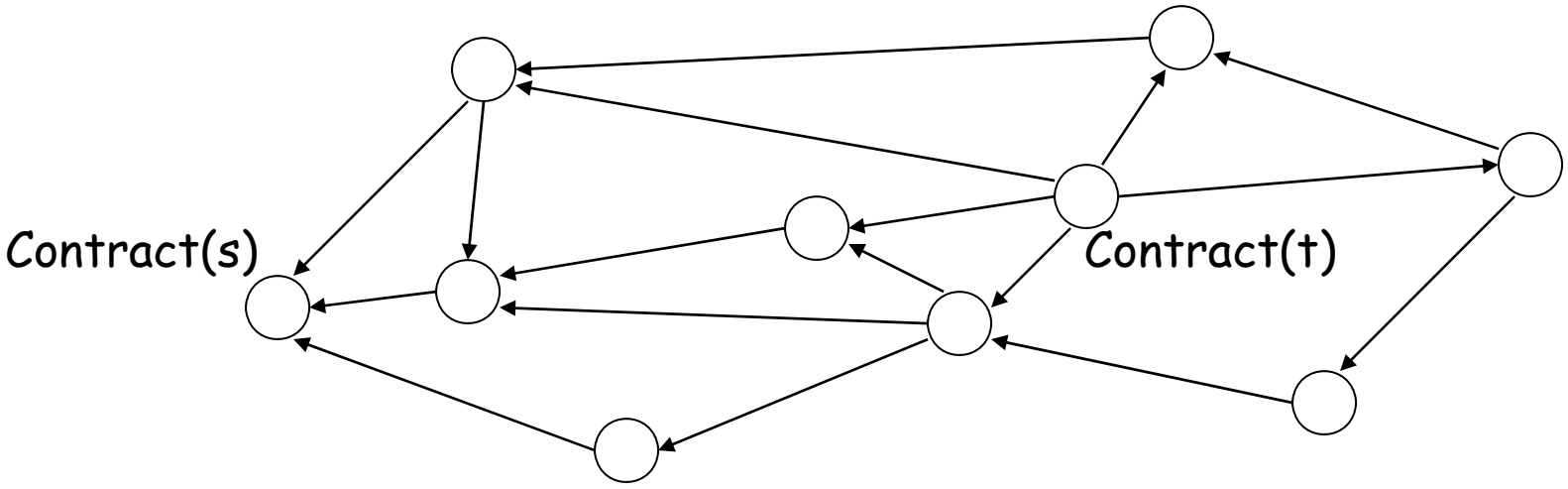

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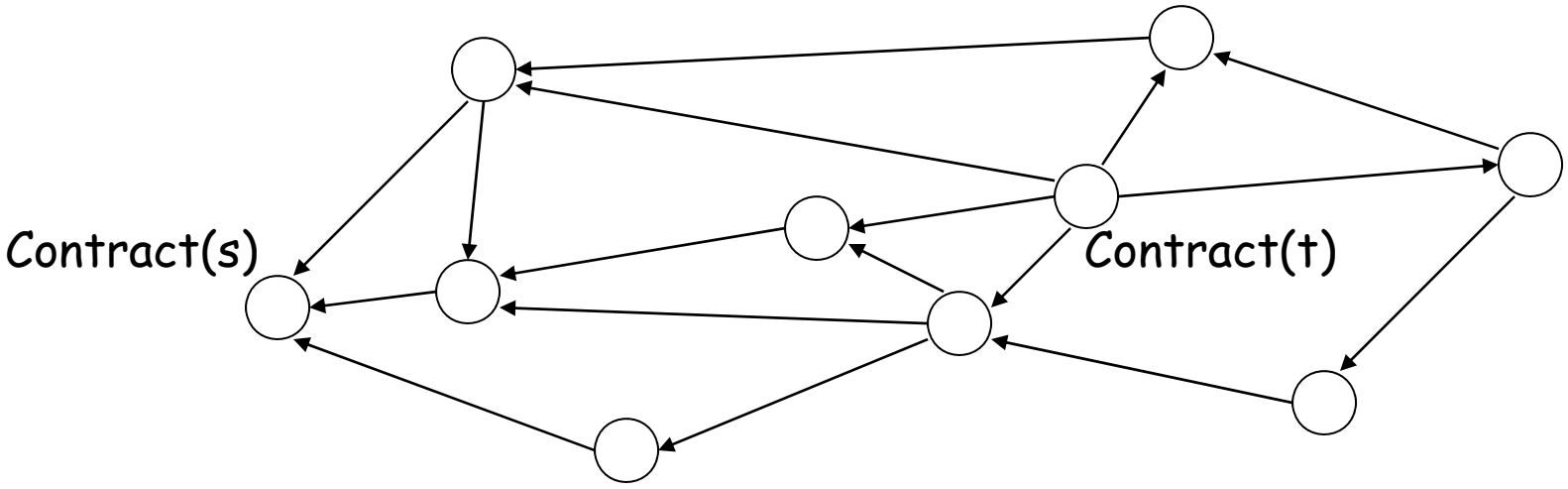
What if s,t are on different faces ? [this paper]

- find contract(t)-contract(s) path, duplicate edges, "merge" edges within the same face
- construct the dual, except no edges cross the new path
- sum # paths between faces sharing a new edge



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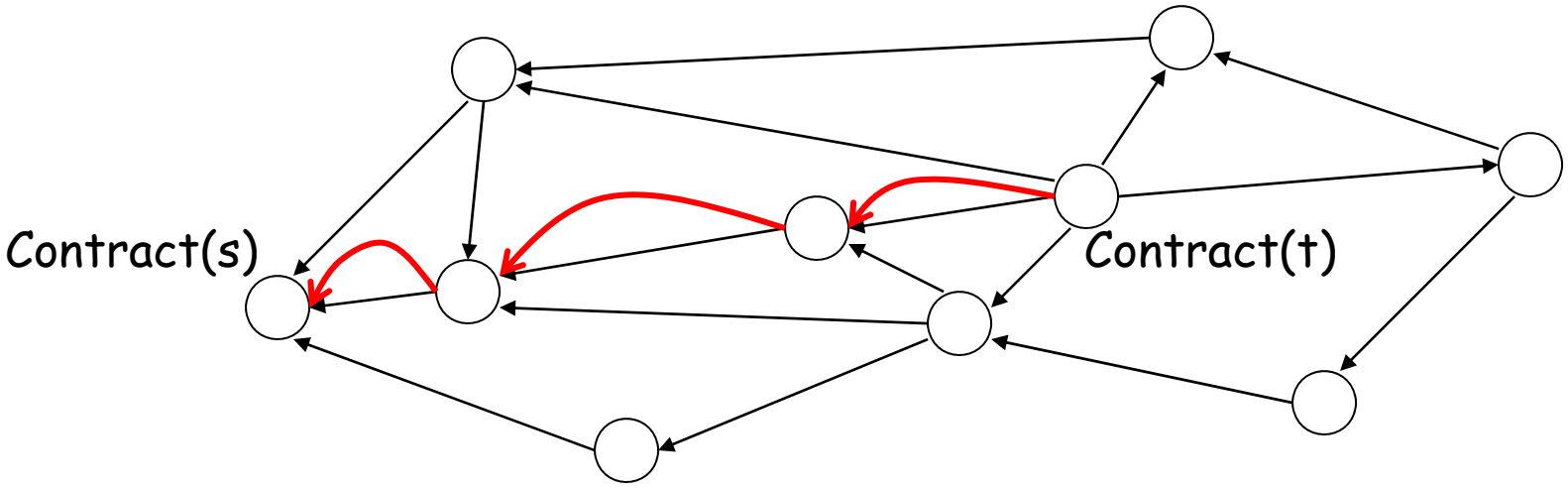
- find contract(t)-contract(s) path, duplicate edges, "merge" edges within the same face
- construct the dual, except no edges cross the new path
- sum # paths between faces sharing a new edge



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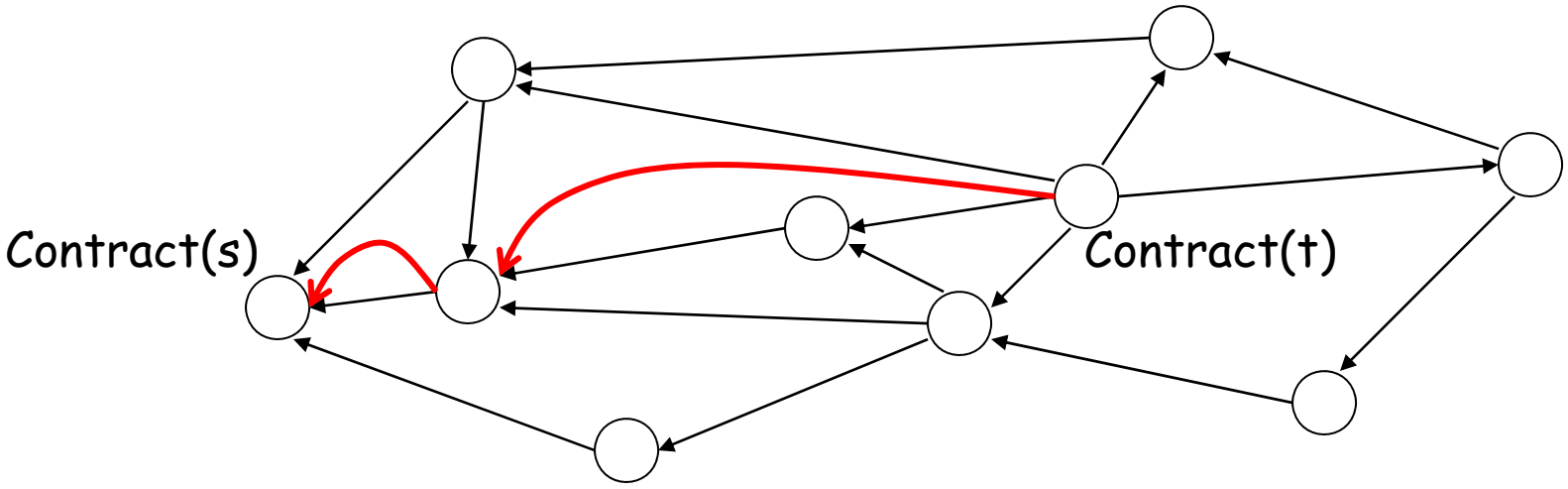

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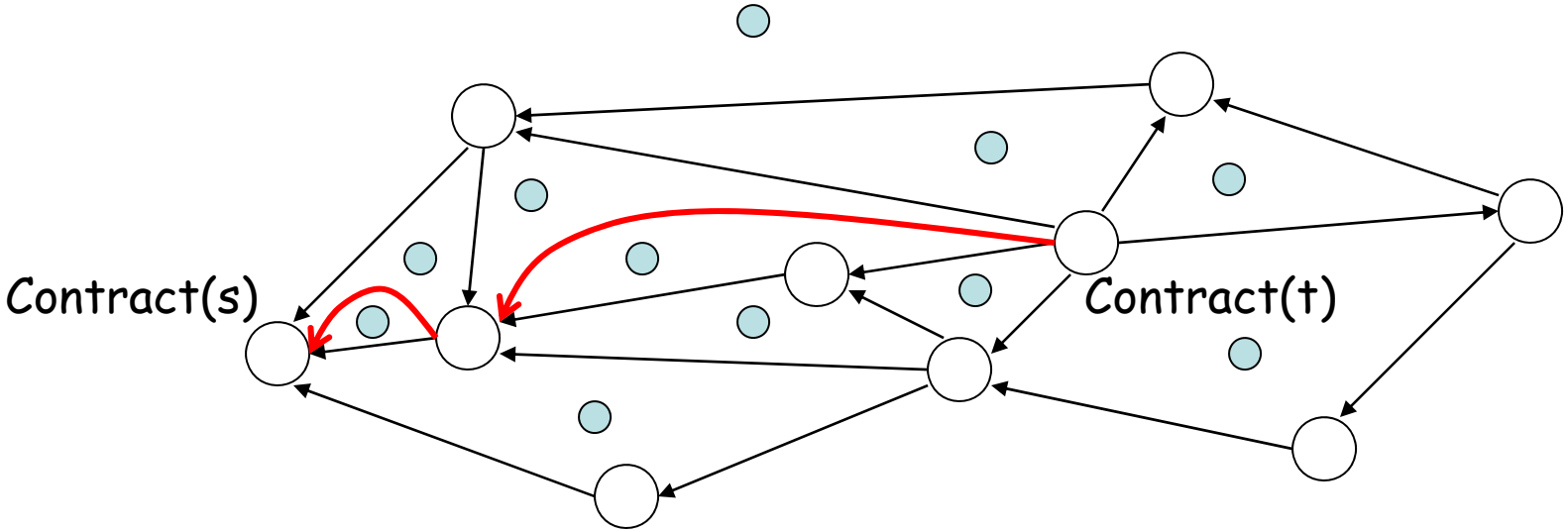

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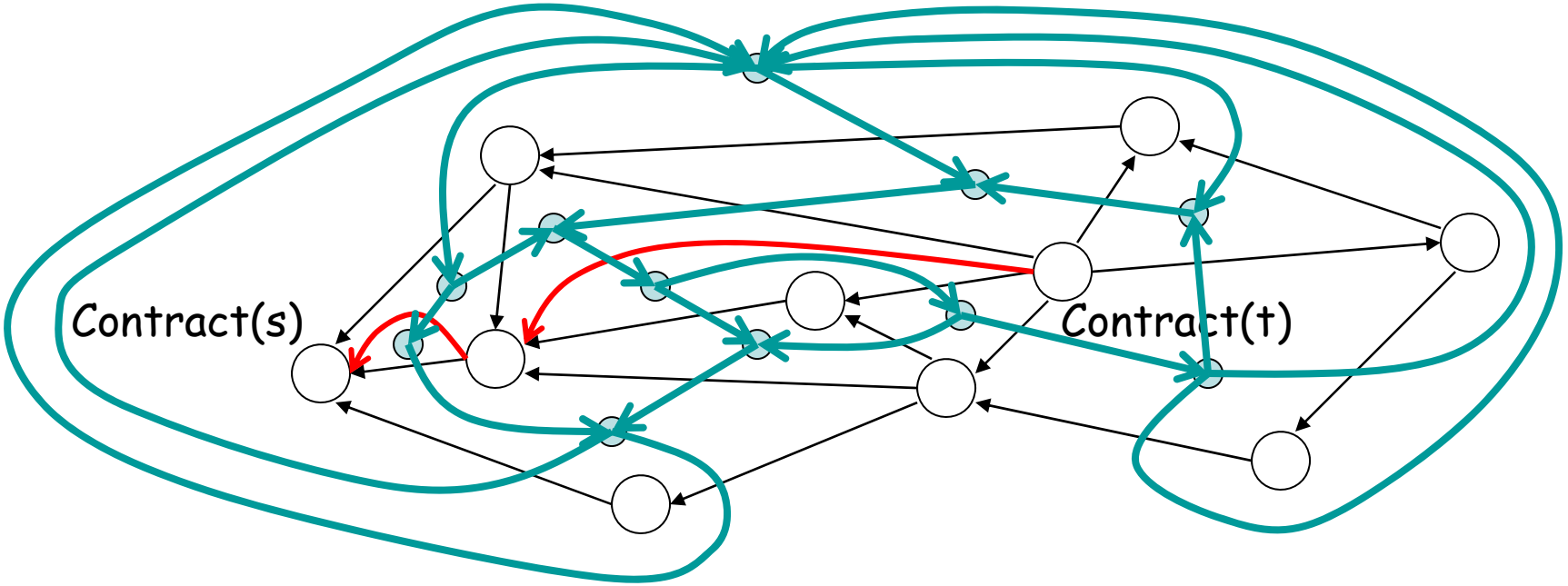

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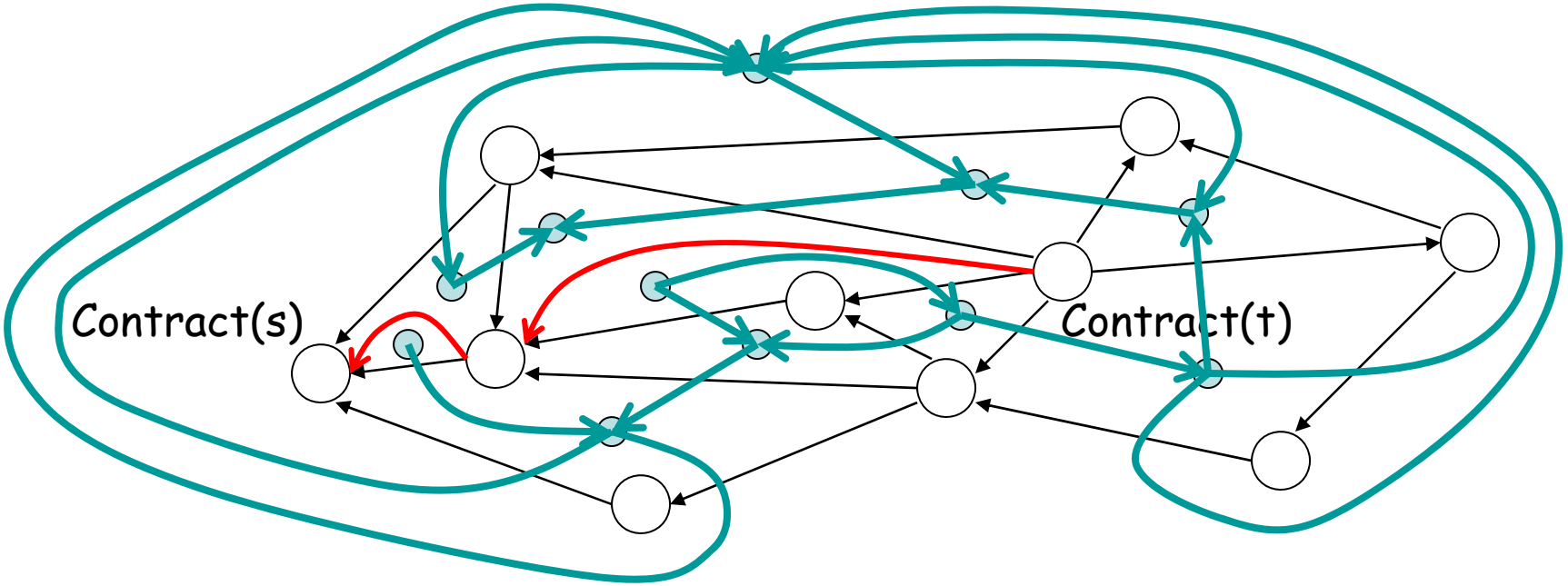
- construct the dual, except no edges cross the new path -> a **DAG**
- sum # paths between faces sharing a new edge



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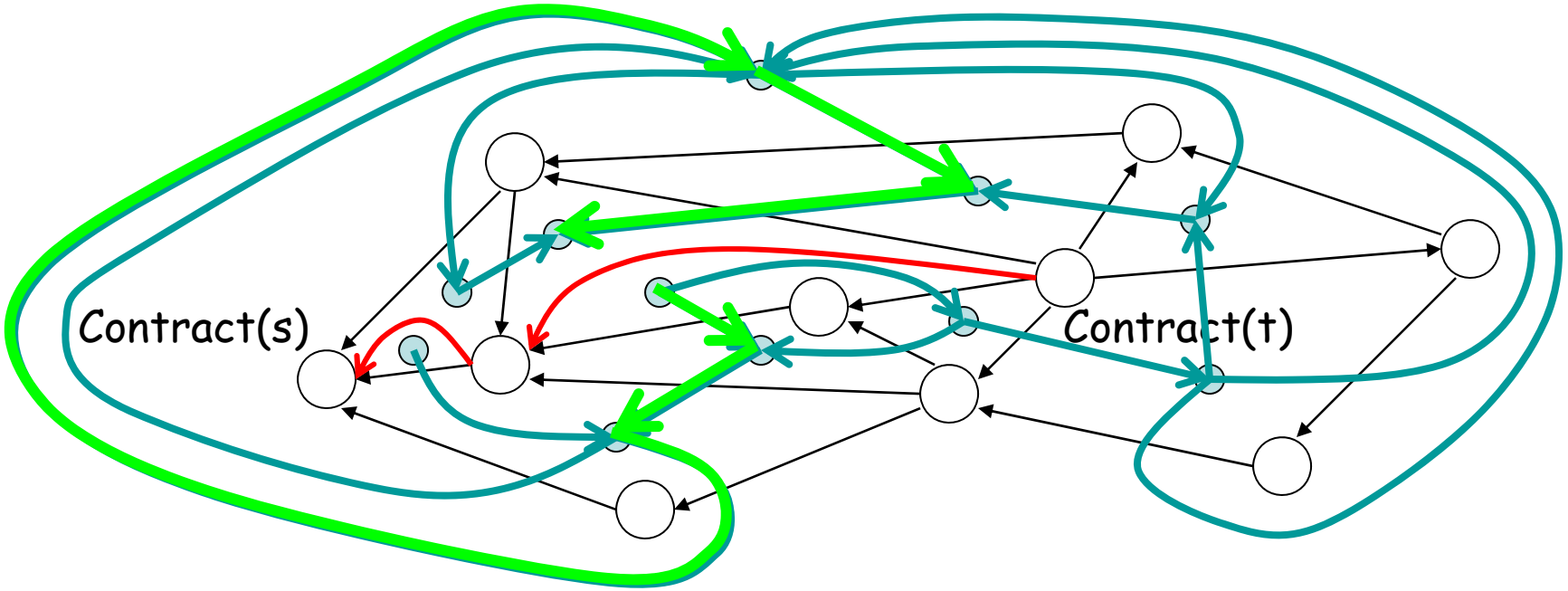

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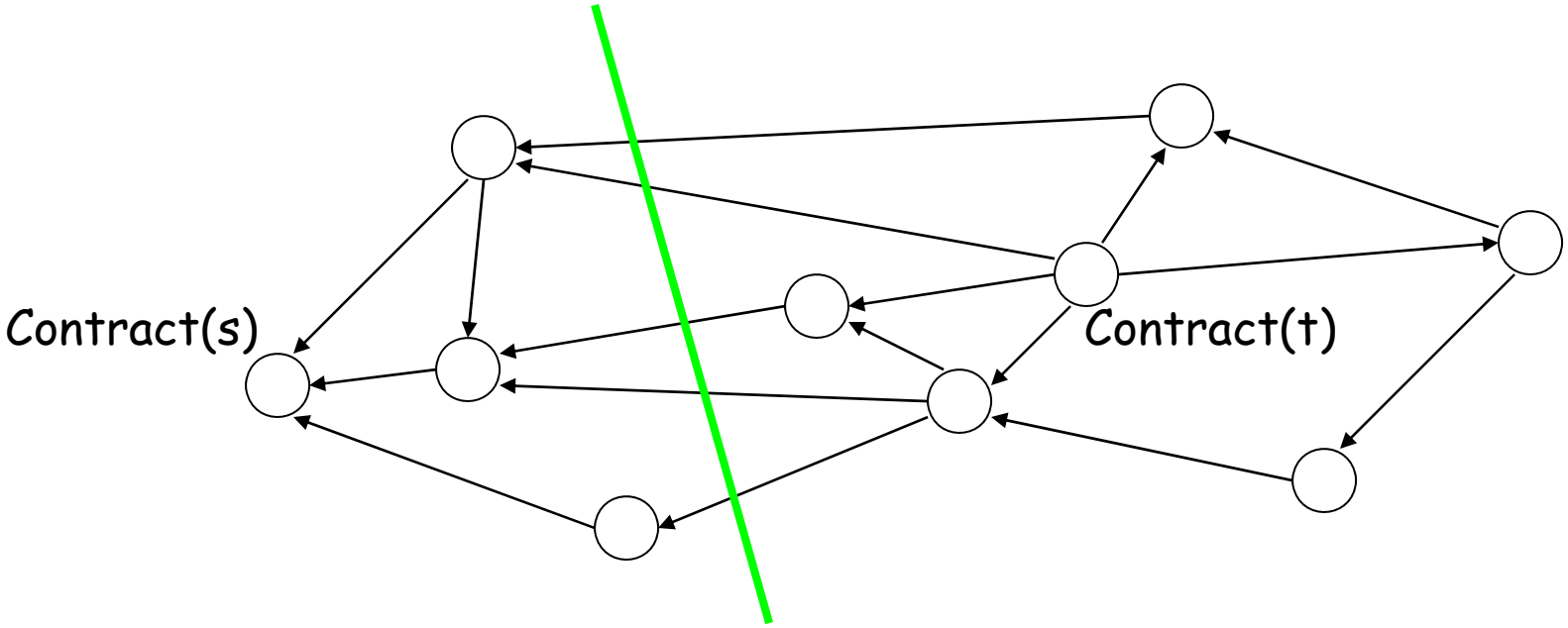
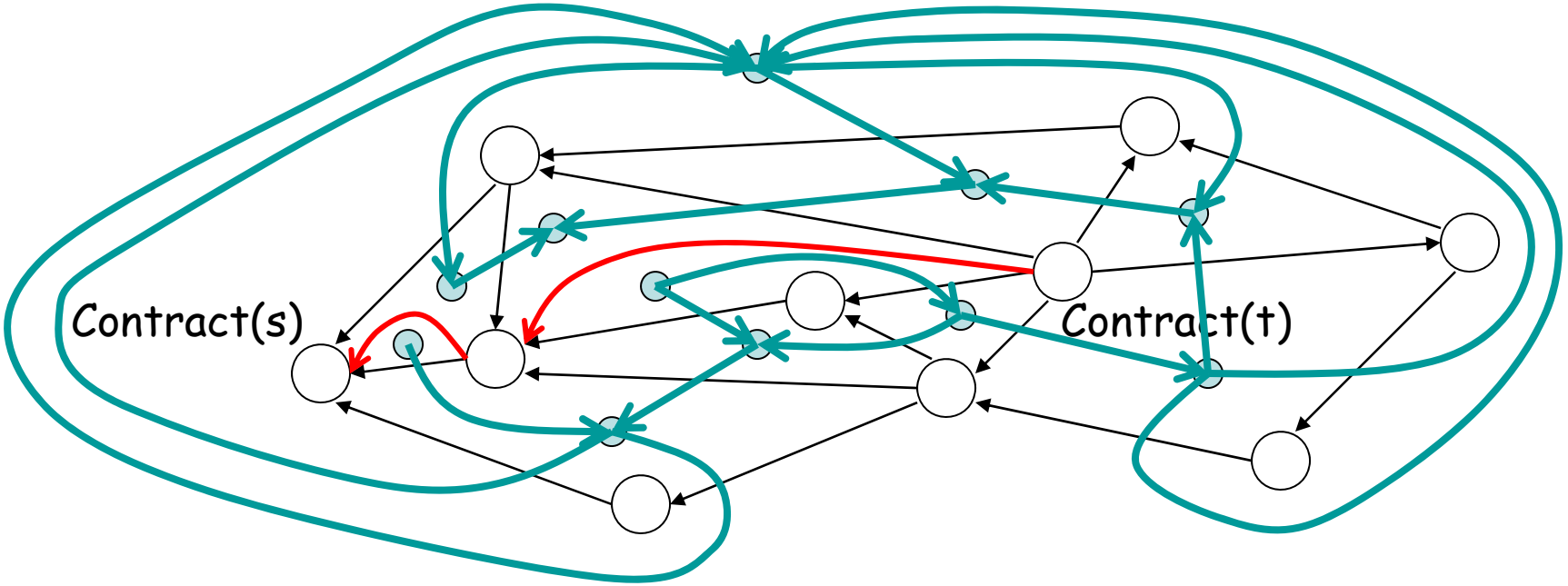


"Forward-cuts" and different faces

Why does it work ?

- the dual (without the cross edges) is a DAG [whenever all vertices lie on an s-t path – a standard assumption]
- a simple dynamic programming counts all paths between two end-points in a DAG



Running time

Reduction to forward cuts:

- $O(n \log n)$ to find a (acyclic) max-flow in planar graphs [Borradaile-Klein '09]
- $O(n)$ to find and contract the strongly connected components

Counting forward cuts:

- $O(n)$ find the path, construct the dual graph
- $O(n)$ compute #paths between two end-points in the dual
- $O(dn)$ overall computation of paths, at most d end-point pairs where d = length of the s-t path

TOTAL: $O(dn + n \log n)$

Open Problems

Many open problems:

- non-planar graphs ? (unweighted or weighted)
- graph arising in computer vision (high-dimensional grids, with extra source/sink vertices)
- computing the sum of all (s,t)-cut weights (i.e., weighted counting = the partition function) -> important for parameter estimation, e.g., in the Markov Random Field model in vision