Perfect Matchings in a Planar Graphs

Recall the definitions of:

- **matchings**
  - given a graph $G$, a matching is a collection of edges of $G$ pairwise non-adjacent.

- **perfect matchings**
  - a matching that covers all vertices (of size $n/2$), e.g., example above.

- **planar graphs**
  - a graph that can be drawn in a plane without edges crossing other edges, e.g., example above.

Recall the (monomer-)dimer problem from physics.

- **atoms** (nodes)
- **possible connections** (edges)
- **dimers**: molecules with 2 atoms
- **monomers**: single atoms (not in a molecule)

$\equiv$ perfect match.
$\equiv$ monomer-dimer config.
Fact 1.9:
Let \( M, M' \) be two perfect matchings, then \( M \cup M' \) forms a collection of single edges and even-length cycles.
Def:

Let $G \rightarrow$ be an orientation of $G$ and let $C$ be a cycle of $G$. $C$ is **oddly oriented** if the odd number of edges in $C$ disagrees with $G \rightarrow$.

$G \rightarrow$ is **Pfaffian** if, for every perfect matchings $M, M'$, every cycle in $M \cup M'$ is oddly oriented.

**Example:** is $G \rightarrow$ Pfaffian?

perf. m. of $G$: 

\[ \begin{array}{c c c}
\text{1} & \text{2} & \text{3} \\
\text{4} & \text{5} & \text{6}
\end{array} \]

$\Rightarrow$ thus, the $G \rightarrow$ above is Pfaffian.

Note: not every cycle in $G$, just every cycle in $M \cup M'$
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Def: Skew adjacency matrix of $G\rightarrow$ is $A_s(G\rightarrow)$ where

$$a_{ij} = \begin{cases} 
+1 & \text{if } (i,j) \text{ is an edge in } G\rightarrow \\
-1 & \text{if } (j,i) \text{ is an edge in } G\rightarrow \\
0 & \text{otherwise}
\end{cases}$$

$A_s(G)$ is a skew adjacency matrix.
Thm 1.11 (Kasteleyn):
For any Pfaffian orientation $G \rightarrow$, 
\[
\# \text{ perfect matchings of } G = \sqrt{\det A_s(G \rightarrow)}
\]

Example:
\[
\begin{vmatrix}
0 & 1 & 0 & 1 \\
-1 & 0 & 1 & -1 \\
0 & -1 & 0 & 1 \\
-1 & 1 & -1 & -1
\end{vmatrix}
\]

\[
\begin{vmatrix}
0 & 1 & 0 & 1 \\
-1 & 0 & 1 & -1 \\
0 & -1 & 0 & 1 \\
-1 & 1 & -1 & -1
\end{vmatrix}
\]

Thm 1.14: Every planar graph has a Pfaffian orientation (and it can be easily constructed).

Note: It is not known whether the problem of deciding if a graph has a Pfaffian orientation is in P nor whether it is NP-complete.
Def: Let $G$ be obtained from $G$ by replacing every edge by $\leftrightarrow$. An even cycle cover of $G$ is a collection of even cycles that cover every vertex exactly once.

Lemma 1.12: There is a bijection between (ordered) pairs of perfect matchings and even cycle covers of $G$. 

Notice: if $X$ is a perfect matching in $G$, then the number of even cycle covers of $G$ is $X^2$.

Thus, if we show that the number of even cycle covers of $G$ is $\det(A_s(G))$, where $G$ is Pfaffian, then it follows that

$$\# \text{ perf. match. in } G = \sqrt{\det(A_s(G))}.$$
Thm 1.11: If $G \rightarrow$ Pfaffian, then

$$(\# \text{ perfect matchings of } G)^2 = \det A_s(G \rightarrow)$$

Proof:
Lemma 1.13: Let $G\to$ be a connected planar digraph (drawn in plane) with all faces except outer having an odd number of clockwise oriented edges. Then, in any (simple) cycle $C$, the number of clockwise edges is opposite parity as the number of vertices inside $C$.

In particular, $G\to$ from Lemma 1.13 is Pfaffian.
Lemma 1.13: Let $G\rightrightarrows$ be a connected planar digraph (drawn in plane) with all faces except outer having an odd number of clockwise oriented edges. Then, in any (simple) cycle $C$, the number of clockwise edges is opposite parity as the number of vertices inside $C$.

Proof:

Let $v = \# \text{ vertices inside } C$

$k = \text{length of } C$

$c = \# \text{ clockwise edges in } C$

$f = \# \text{ faces inside } C$

$e = \# \text{ edges inside } C$

$c_i = \# \text{ clockwise edges in component } i \text{ inside } C$
Pfaffian orientations in planar graphs

Thm 1.14: Every planar graph has a Pfaffian orientation.

Proof: