380 — Sample Midterm

Answers and some notes. I typed this up quickly; let me know if you find any mistakes.

- This exam consists of five questions.
- Each question is worth 10 points.
- The exam is worth 40 points.
- Your exam score is the sum of your four highest question scores.
- Calculators etc. are not allowed.
1. Let $\Sigma = \{0, 1\}$. In this question, every string over $\Sigma$ is viewed as a binary natural number. Leading 0s are allowed. For example, 00100 is viewed as the number 4. $\Lambda$ is viewed as the number 0. Look at the following list of six languages.

- $L_1 = \{ x \in \Sigma^* \mid x \geq 16 \}$.
- $L_2 = \{ x \in \Sigma^* \mid x$ is a square and $x < 1000 \}$.
- $L_3 = \{ x \in \Sigma^* \mid x$ is not divisible by 3 \}.
- $L_4 = \{ x \in \Sigma^* \mid x$ is divisible by 666 \}.
- $L_5 = \{ xy \mid x, y \in \Sigma^* \text{ and } x \cdot y = 12 \}$.
- $L_6 = \{ xy \mid x, y \in \Sigma^*, |x| = |y|, \text{ and } x \text{ and } y \text{ represent the same number} \}$.

(a) Which of the languages from the list are regular?
- $L_1$, $L_2$, $L_3$, $L_4$, and $L_5$.

(b) Which of the languages from the list contain $\Lambda$?

(c) Draw the transition diagram of a FA that accepts one of the languages from the list or give a regular expression that represents one of the languages from the list. Also state which language you chose.

_Different answers are possible. You want to pick a case that is easy and that you are sure about. Also you should do just one case!_

- **Regular expression for $L_1$**:
  
  $$0^1(0 + 1)^*(0 + 1)^4$$

  The FA for $L_1$ is also easy.

- **$L_2$, $L_4$, and $L_6$ is pretty easy as well. Regular expression**:
  
  $$0^*10^*1100 + 0^*100^*110 + 0^*110^*100 + 0^*1000^*11 + 0^*1100^*10 + 0^*11000^*1$$

  The regular expression for $L_4$ is easy, but long. You need to “+” together all squares less than 1000 in binary and precede that by $0^*$, since leading 0s are allowed.
2. Let $M$ be the following NFA-A. $M = (Q, \Sigma, q_0, A, \delta)$ such that $Q = \{1, 2, 3, 4\}$, $\Sigma = \{a, b\}$, $q_0 = 1$, $A = \{3\}$, and $\delta$ is given by the following transition table:

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\delta(q, A)$</th>
<th>$\delta(q, a)$</th>
<th>$\delta(q, b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${2}$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>2</td>
<td>${3}$</td>
<td>$\emptyset$</td>
<td>${2}$</td>
</tr>
<tr>
<td>3</td>
<td>$\emptyset$</td>
<td>${3, 4}$</td>
<td>${1}$</td>
</tr>
<tr>
<td>4</td>
<td>$\emptyset$</td>
<td>${2}$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

(a) Draw the transition diagram of $M$.

*I assume that all of you are capable of doing this.*

(b) The book describes how to transform an NFA-A into an equivalent NFA. Draw the transition diagram of the equivalent NFA that is produced by applying this algorithm on $M$. Do not simplify the NFA.

*I will give the quintuple definition; you can convert this to the transition diagram.*

$M = (Q, \Sigma, q_0, A, \delta)$ such that $Q = \{1, 2, 3, 4\}$, $\Sigma = \{a, b\}$, $q_0 = 1$, $A = \{1, 3\}$, and $\delta$ is given by the following transition table:

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<th>$\delta(q, a)$</th>
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<tbody>
<tr>
<td>1</td>
<td>${3, 4}$</td>
<td>${1, 2, 3}$</td>
</tr>
<tr>
<td>2</td>
<td>${3, 4}$</td>
<td>${1, 2, 3}$</td>
</tr>
<tr>
<td>3</td>
<td>${3, 4}$</td>
<td>${1, 2, 3}$</td>
</tr>
<tr>
<td>4</td>
<td>${2, 3}$</td>
<td>$\emptyset$</td>
</tr>
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</table>

(c) Give a regular expression that is equivalent to $L(M)$. Make your expression as simple as possible.

*Answer: $(a + b)^*$
3. Let $\Sigma = \{a, b\}$. Let $\text{double}$ be the function from $\Sigma^*$ to $\Sigma^*$ that doubles each character in a string. For example, $\text{double}(baaba) = bbbaaabbaa$.

(a) What is $\text{double}(aabaa)$?
   $\text{aaaabbaaaa}$

(b) What is $\text{double}(\Lambda)$?
   $\text{\Lambda}$

(c) Suppose $x \in \{a, b\}^*$ and the length of $x$ is $k$. What is the length of $\text{double}(x)$?
   $2k$

(d) Give a recursive definition of function $\text{double}$ from $\Sigma^*$ to $\Sigma^*$.
   - $\text{double}(\Lambda) = \Lambda$.
   - For all $x \in \Sigma^*$ and $\sigma \in \Sigma$, $\text{double}(x\sigma) = \text{double}(x)\sigma\sigma$.

For $L$ a language over $\Sigma$, define $\text{double}(L)$ as follows:

$$\text{double}(L) = \{ \text{double}(x) \mid x \in L \}.$$  

(e) Let $A$ be the language $\{ab, bbb, baba\}$. What is $\text{double}(A)$?
   $\{aabb, bbbbbb, bbaabbaa\}$

(f) List all languages $B$ over $\Sigma$ that have the property that $\text{double}(B) = B$.
   $\emptyset$ and $\{\Lambda\}$

(g) For each of the following statements, circle the right answer.
   i. If $L$ is a regular language over $\Sigma$, then $\text{double}(L)$ is regular. True
   ii. If $L$ is a finite language over $\Sigma$, then $\text{double}(L)$ is finite. True
   iii. If $L \subseteq \Sigma^*$ is not regular, then $\text{double}(L)$ is not regular. True
   iv. If $L_1$ and $L_2$ are regular languages over $\Sigma$, then $L_1 \cup L_2$ is regular. True
   v. If $L_1$ and $L_2$ are finite languages over $\Sigma$, then $L_1 \cup L_2$ is finite. True
   vi. If $L_1 \subseteq \Sigma^*$ and $L_2 \subseteq \Sigma^*$ are not regular, then $L_1 \cup L_2$ is not regular. False
4. Let $L = \{a^ib^k a^\ell \mid \ell > i + k\}$.

(a) List all strings in $L$ of length 7.

$a^7, ba^6, aba^5, bba^5, aaba^4, abba^4, b^3a^4$

(b) Use the Pumping Lemma for Regular Languages to prove that $L$ is not regular.

Suppose for a contradiction that $L$ is regular. Let $n$ be the integer from the pumping lemma.

Let $x = b^n a^{n+1}$. Then $x \in L$ and $|x| \geq n$. By the pumping lemma, there exist strings $u, v, w$ such that $x = uvw$, $|uv| \leq n$, $|v| > 0$, and $uv^mw \in L$ for all $m \geq 0$.

In particular, $uv^2w \in L$. (Choose $m = 2$.)

Since $|uv| \leq n$, $uv^2w = b^{n+|v|}a^{n+1}$. Since $|v| > 0$, $n + 1 \nless n + |v|$. It follows that $uv^2w \notin L$. This is a contradiction.

It follows that the assumption that $L$ is regular is wrong.

So, we have shown that $L$ is not regular.
5. This question is about the subset construction. In this question, take $\Sigma = \{a, b, c\}$.

(a) If you literally apply the subset construction to an NFA with $k$ states, how many states will the corresponding FA have? $2^k$

(b) Give a simple example of a minimal NFA such that the FA obtained by literally applying the subset construction is not minimal. (A minimal NFA is an NFA such that no equivalent NFA has fewer states.) For your answer, draw the transition diagram of the NFA, the transition diagram of the FA obtained by applying the subset construction, and briefly argue that the FA is not minimal.

Consider the NFA $M_1$ with one rejecting state and no transitions (you need to draw the transition diagram). Clearly, $M_1$ is minimal, since an NFA can not have fewer than one state. $L(M_1) = \emptyset$ and there exists a one-state FA that accepts $\emptyset$. But the subset construction gives a two-state FA (you need to draw the transition diagram).

(c) Give a simple example of a minimal NFA such that the FA obtained by literally applying the subset construction is minimal. For your answer, draw the transition diagram of the NFA, the transition diagram of the FA obtained by applying the subset construction, and briefly argue that the FA is minimal.

Consider the NFA $M_2$ with one accepting state and no transitions (draw it). Clearly, $M_2$ is minimal, since an NFA can not have fewer than one state. $L(M_2) = \{\Lambda\}$. The two-state FA given by the subset construction (draw it) is minimal, since an FA that accepts $\Lambda$ needs at least one accepting state, and an FA that does not accept $\Sigma^*$ needs at least one rejecting state.