For completeness, recall that we have defined our complete FA, $M_1$, as follows: $M_1 = (Q \cup \{q_D\}, \Sigma, q_0, A, \delta_1)$, where

$$\forall q \in Q_1, \forall \sigma \in \Sigma, \delta_1(q, \sigma) = \begin{cases} \delta(q, \sigma) & \text{if } \delta(q, \sigma) \text{ is defined}, \\ q_D & \text{otherwise}. \end{cases}$$

We will prove that $L(M_1) = L(M)$ by proving that

$$\forall x \in \Sigma^*, \delta_1^*(q_0, x) = \begin{cases} \delta^*(q_0, x) & \text{if } \delta^*(q_0, x) \text{ is defined}, \\ q_D & \text{otherwise}. \end{cases}$$

We proceed by structural induction on $x$.

**Basis** Let $x = \Lambda$.

$$\delta_1^*(q_0, \Lambda) = q_0 = \delta^*(q_0, \Lambda)$$

**Inductive Hypothesis** $x$ is a string such that

$$\delta_1^*(q_0, x) = \begin{cases} \delta^*(q_0, x) & \text{if } \delta^*(q_0, x) \text{ is defined}, \\ q_D & \text{otherwise}. \end{cases}$$

**Inductive Step** We must now show that $\forall \sigma \in \Sigma,$

$$\delta_1^*(q_0, x\sigma) = \begin{cases} \delta^*(q_0, x\sigma) & \text{if } \delta^*(q_0, x\sigma) \text{ is defined}, \\ q_D & \text{otherwise}. \end{cases}$$

If $\delta^*(q_0, x\sigma)$ is defined, then $\delta^*(q_0, x)$ is also defined, so we proceed as follows:

$$\delta_1^*(q_0, x\sigma) = \delta_1^*(q_0, x), \sigma) = \delta_1(\delta^*(q_0, x), \sigma) = \delta(\delta^*(q_0, x), \sigma) = \delta^*(q_0, x\sigma)$$

(by definition of $\delta_1^*$)

(by Inductive Hypothesis)

(by definition of $\delta_1$)

(by definition of $\delta^*$)

If $\delta^*(q_0, x\sigma)$ is not defined, then either $\delta^*(q_0, x)$ is not defined, or $\delta(\delta^*(q_0, x), \sigma)$ is not defined. In the first case,

$$\delta_1^*(q_0, x\sigma) = \delta_1(\delta_1^*(q_0, x), \sigma) = \delta_1(q_D, \sigma) = q_D$$

(by definition of $\delta_1^*$)

(by Inductive Hypothesis)

(by definition of $\delta_1$)

In the second case,

$$\delta_1^*(q_0, x\sigma) = \delta_1(\delta_1^*(q_0, x), \sigma) = \delta_1(\delta^*(q_0, x), \sigma) = \delta(\delta^*(q_0, x), \sigma)$$

(by definition of $\delta_1^*$)

(by Inductive Hypothesis)

(by definition of $\delta_1$)
But, $\delta(\delta^*(q_0, x), \sigma)$ is not defined, so $\delta_1(\delta^*(q_0, x), \sigma) = q_D$. Thus,

$$
\delta^*_1(q_0, x\sigma) = \begin{cases} 
\delta^*(q_0, x\sigma) & \text{if } \delta^*(q_0, x\sigma) \text{ is defined}, \\
q_D & \text{otherwise.}
\end{cases}
$$

We have shown that when $\delta^*(q_0, x)$ is defined, $\delta^*_1(q_0, x) = \delta^*(q_0, x)$. Since $M$ and $M_1$ share the same set of accepting states, we may conclude that $M$ accepts $x \in \Sigma^*$ if and only if $M_1$ accepts $x$. Furthermore, we have shown that when $\delta^*(x, q_0)$ is not defined, $\delta^*_1(x, q_0) = q_D$. Since $q_D \notin A$, and since, by construction, $\delta_1(q_D, \sigma) = q_D, \forall \sigma \in \Sigma$, we may conclude that $M$ rejects $x \in \Sigma^*$ if and only if $M_1$ rejects $x$. Therefore, $L(M_1) = L(M)$. 

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