NP-Completeness

Thm 7.27 [Cook-Levin]: SAT is in P iff $P = NP$. 

\[ P = NP \quad \text{iff} \quad P \subseteq NP \]
Def 7.29: Language $A$ is **polynomial-time reducible** to language $B$, written $A \leq_p B$, if a polynomial-time computable function $f: \Sigma^* \rightarrow \Sigma^*$ exists such that for every $w$,

$$w \in A \iff f(w) \in B$$

The function $f$ is called **polynomial-time reduction** of $A$ to $B$.

Thm 7.31: If $A \leq_p B$ and $B \in P$, then $A \in P$. 

*overall run. time of solver of $A$: $O(n^{dc})$*
NP-Completeness

Thm 7.32: 3SAT is polynomial-time reducible to CLIQUE, where

\[ 3\text{SAT} = \{ \langle \phi \rangle \mid \phi \text{ is a satisfiable } 3\text{-cnf formula} \} \]

Each "clause" has 3 literals (permittive);
- clauses are joined by \( \wedge \);
- within clauses only \( \lor \).

Draw edges between literals from different clauses, except between \( x_i \) and \( \overline{x_i} \).

If \( \|G\| = n \)
- then \( G \) has \( \leq n \) vertices
- \( \leq n^2 \) edges

Hence, \( G \) is \( p\)-time.

\( k \)-SAT reduction:
- If \( \phi \) has a solution;
  \( f(\phi) = \{ 0,1,2 \} \)
- \( f(G) \) has a \( k \)-clique.
Def 7.34: A language B is **NP-complete** if it satisfies both conditions:

- B is in NP, and

- every A in NP is polynomial-time reducible to B.
**Def 7.34:** A language $B$ is **NP-complete** if it satisfies both conditions:
- $B$ is in NP, and
- every $A$ in NP is polynomial-time reducible to $B$.

**Thm 7.35:** If $B$ is NP-complete and $B \in P$, then $P = NP$. 
Def 7.34: A language $B$ is **NP-complete** if it satisfies both conditions:

- $B$ is in NP, and
- every $A$ in NP is polynomial-time reducible to $B$.

Thm 7.36: If $B$ is NP-complete and $B \leq_p C$ for some $C \in$ NP, then $C$ is NP-complete.
Def 7.34: A language $B$ is **NP-complete** if it satisfies both conditions:
- $B$ is in $NP$, and
- every $A$ in $NP$ is polynomial-time reducible to $B$.

Thm 7.37 [Cook-Levin]: SAT is NP-complete.

Proof idea: take a NP problem, say $A$. need to provide $f$ s.t. $WEA \iff f(w) \in SAT$

$f$ encodes the computation of the NTM for $A$ into a satisfiable formula $\phi$

somewhat similar to the reduction we did for PCP, check the book for details.

we showed $SAT \leq_p CLIQUE$ (and last class $CLIQUE \in NP$) $\Rightarrow$ CLIQUE is NP-c.

Note: a long list of known NP-complete problems.