The Halting Problem

\[ A_{TM} = \{ <M,w> \mid M \text{ is a TM that accepts string } w \} \]

- Turing-recognizable?  
  - just execute \( M \) on \( w \)
- Turing-decidable?  
  - not
A bit about infinite sets and their sizes (diagonalization):

**Def 4.12:** Let \( A, B \) be sets and let \( f: A \to B \). We say that \( f \) is
- **one-to-one** if \( f(a) \neq f(b) \) for every \( a \neq b \)
- **onto** if for every \( b \in B \) there exists \( a \in A \) such that \( f(a) = b \)

If \( f \) is one-to-one and onto, then \( A, B \) are the **same size** and \( f \) is called **correspondence**.

**Example:** \( \mathcal{N} = \{1, 2, 3, 4, 5, \ldots\} \) and \( \{2, 4, 6, 8, \ldots\} \)
\[
f(x) = 2^x
\]
A bit about infinite sets and their sizes (diagonalization):

Def 4.12: Let $A, B$ be sets and let $f: A \rightarrow B$. We say that $f$ is
- one-to-one if $f(a) \neq f(b)$ for every $a \neq b$
- onto if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$

If $f$ is one-to-one and onto, then $A, B$ are the same size and $f$ is called correspondence.

Example: \( \mathbb{N} = \{1, 2, 3, 4, 5, \ldots\} \) and \{2, 4, 6, 8, \ldots\}

Def 4.14: A set is countable if it is finite or has the same size as \( \mathbb{N} \).
The Halting Problem

Are \( \mathbb{Q} \) (rational numbers) and \( \mathbb{R} \) (real numbers) countable?

Suppose \( \mathbb{R} \) is countable, then we have

\[
\begin{align*}
\text{f: } & \mathbb{N} \rightarrow \mathbb{R} \\
\text{f(1)} &= 0.0\overline{3269} \\
\text{f(2)} &= 1.20111 \\
\text{f(3)} &= 5.000\overline{59} \\
\end{align*}
\]

then can construct a real number that nobody maps to

in the i-th number take a digit that is different from the i-th decimal digit in \( f(i) \)

\( \Box \)
Cor 4.18: There is a language that is not Turing-recognizable.

Pf: CLAIM 1: the set of all T-recognizable languages is countable
- each of \( \mathcal{T} \) has a TM
- each TM can be encoded in a binary string
- the set of all strings \( \{0,1\}^\ast \) is countable
  \( \rightarrow \) the lexicographical ordering defines \( f \) \( \Box \)

CLAIM 2: the set of all languages is uncountable \( \mathcal{P}(\{0,1\}^\ast) \)

suppose it's countable: \( f(1) = S_1 = \{0,1\} = 0010000000 \ldots \)
\( f(2) = S_2 = \{000,101,1^\ast\} = 101001111 \ldots \)
\( f(3) = S_3 = \{\varepsilon,1\} = 1010000000 \ldots \)

diagonalization: construct a set not mapped by \( f \)
\[ \{ \varepsilon, 0, 1, \ldots \} \]

\( \Box \)
Thm 4.11: $A_{TM}$ is not decidable.

Recall: $A_{TM} = \{ <M,w> \mid M \text{ is a TM that accepts string } w \}$

Pf: suppose it is decidable \iff there exists a TM $H$ for $A_{TM}$ that always halts (accepts/rejects)

$$H(<M,w>) = \begin{cases} \text{accepts if } M(w) \text{ accepts} \\ \text{rejects if } M(w) \text{ does not accept} \end{cases}$$

will construct $G$:

$$G(x) = \begin{cases} \text{run } H \text{ on } <x,<x>> \text{ and then it does the opposite of } H \\ \text{if } x \text{ is not an encoding of a TM, reject} \end{cases}$$

$$G(x) = \begin{cases} \text{if } H(x,x) \text{ accepts } \iff x(x) \text{ accepts} , \text{ then } G \text{ rejects} \\ \text{if } H(x,x) \text{ rejects } \iff x(x) \text{ not accepting, then } G \text{ accepts} \end{cases}$$

$$G(G) = \begin{cases} \text{accepts if } G(G) \text{ does not accept} \\ \text{rejects if } G(G) \text{ accepts} \end{cases}$$

\[\square\]
The Halting Problem

Thm 4.22: A language $L$ is decidable iff $L$ is Turing-recognizable and $\overline{L}$ is Turing-recognizable (we say that $L$ is co-Turing-recognizable).

**Pf:**

1. If $L$ is decidable, then $L$ is Turing-recognizable by def.
2. Let $T$ be a Turing machine for $L$.
3. Construct a Turing machine $T'$ by switching the accept/reject states of $T$.
4. For any input $x$, $T'$ accepts $x$ if $T$ rejects $x$ and $T'$ accepts $x$ if $T$ accepts $x$.

Cor 4.23: $A_{TM}^c$ is not Turing-recognizable.