Nonregular languages

Which of these languages are regular?

- \( B = \{ 0^n1^n \mid n \geq 0 \} \)
- \( C = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 0's and 1's} \} \)
- \( D = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 01's and 10's as substrings} \} \)

all languages feel like “need to count” and thus not regular
(not a formal argument
\( \Rightarrow \) will do this week)

but actually \( D \) is regular \( \Rightarrow \) try to give a FA for \( D \)
Nonregular languages

Which of these languages are regular?

- $B = \{ 0^n1^n \mid n \geq 0 \}$
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Proof by closure properties: [not in the book]

- assume $B$ is non-regular — we’ll use this to show that $C$ is not regular
  - by contradiction: assume $C$ is regular
    - language $0^*1^*$ — regular
    - we know regular languages closed under $\cap$: $C \cap 0^*1^*$ must be regular
      - $= B$ — contradiction: $C$ is non-reg.
Suppose we have a DFA with \( p \) states.

Suppose there is a string of length \( > p \) that is accepted. Are there other strings that are accepted?

\[
|q| = p
\]

\( q \) - string that is accepted

\[
|q| > p
\]

Since \( |q| > p \), while computing on \( q \), we are going to repeat a state.

Let \( kq = kq' \) then \( q, q', \ldots \) are all accepted.
Thm 1.70 [pumping lemma]:

Let $A$ be a regular language. Then there exists a number $p$ s.t. for every string $s \in A$ of length $\geq p$ there exist strings $x, y, z$ s.t.

0. $s = xyz$,

1. For each $i \geq 0$, $xy^i z \in A$,

2. $|y| > 0$, and

3. $|xy| \leq p$. 

In particular:

Want to prove $B$ is not regular.

(Want to say p.l. does not hold.)

For any $p$, $s \in B$ of length $\geq p$ st.

$|xy^i z| = xy^iz$, $|y| > 0$, $|xy^i + p$ st.

there exists $i$ st.

$xy^i z \notin B$
Pumping lemma for regular lang.

Example: \( B = \{ 0^n1^n \mid n \geq 0 \} \)

Prove \( B \) non-regular:

Suppose \( B \) is regular, then the p.l. holds and there exists a \( p \).

Then, take \( s = 0^p1^p \)

\[ x \underbrace{0 \ldots 0}_{p} \underbrace{1 \ldots 1}_{p} \underbrace{y \ldots y}_{y \text{ has at least one } 0} \underbrace{z \ldots z}_{z \text{ has } p 	ext{ ones}} \]

by 3) \( xy \) contain only 0's

by 2) \( y \) has at least one 0

by 1) take \( i = 2 \) then \( xy^2z \) should be in \( B \)

\( xz = 0^p \text{ ones} \)

\( = p + |y| \text{ zeros} > p \)

Therefore, \( xy^2z \not\in B \)

\( B \) not regular \( \square \)
Pumping lemma for regular lang.

Example: \( C = \{ w \mid w \text{ has equal number of 0's and 1's} \} \)

\textbf{Pf:} suppose \( C \) is regular, then the p.l. holds and there is a \( p \).

consider \( s = 0^p1^p \) (notice \( s \in C, |s| = 2p \geq p \))

do the same as previous slide \( \square \)
Example: $F = \{ \text{ww} \mid w \in \{0,1\}^* \}$

**Pf:** Assume $F$ is regular, then let $p$ be the p.l. constant. Consider $s = 0^p1^p0^p1^p$ (notice $s \in F$ and $|s| > 2p$).

Take $i = 0$ by 1) $xy^0z = xz \notin F$ where $y = 0^p1^p0^p1^p \notin F$ since $|y| > 0$.
Pumping lemma for regular lang.

Example: \( D = \{ 1^k \mid k \geq 0 \text{ is a square} \} \)

\( D \) is non-reg. \( \xleftarrow{\text{want to prove}} \)

Suppose \( D \) is regular. Let \( p^2 \) be the p.l. constant.

Consider \( s = 1^{p^2} \)

Suppose \( xy^2z = s \) and conditions 1-3 hold

\( \rightarrow \)

by 2 \( 1y| > 0 \) \( \bigg\} \) let \( 1y| = \delta \)

by 3 \( 1y| \leq p \)

by 1 take \( i = 0 \)

\( xy^i z = 1^{p^2} + (i-1)\delta = 1^{p^2-\delta} \)

then \( p^2 - p \leq |xy^i z| \leq p^2 - 1 \)

the closest smaller square to \( p^2 \): \( (p-1)^2 = p^2 - 2p + 1 \)

therefore there is no square in \( \{ p^2-p, \ldots, p^2-1 \} \)

and thus \( xy^i z \notin D \quad \square \)
Pumping lemma for regular lang.

Section 1.4

Example: \( E = \{ 0^i1^j \mid i > j \} \)

Take \( s = 0^{p+1}1^p \)

Within 0's

\( l_y > 0 \)

Take \( i = 0 \)

\( xy^0z \) contains \( p \) ones

\( p+1-l_y \) zeros

\( \notin E \)

\( \square \)