Determinism: computation always continues in a uniquely determined way.

Nondeterminism: have more (or none) choices

Example:

\[ \{ w \in \{0,1\}^* \mid w \text{ contains 001 or 0101 as a substring} \} \]
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Example:

\[ \{ w \in \{0,1\}^* \mid w \text{ contains } 001 \text{ or } 0101 \text{ as a substring} \} \]

Nondeterministic FA can also use \( \varepsilon \)-transitions:
Example:

\[ \{ w \in \{0,1\}^* \mid w \text{ contains 1 in the third position from the end} \} \]

Does there exist a (deterministic) FA recognizing this language?
Nondeterminism

Example:
\{ w \in \{0,1\}^* \mid w \text{ contains 1 in the third position from the end} \}

Does there exist a (deterministic) FA recognizing this language?
Example:

\[ \{ w \in \{0\}^* \mid |w| \text{ is divisible by 2 or 3} \} = A \]
**Nondeterminism**

Formal definition:

A *nondeterministic finite automaton* (NFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- $Q$ is a finite set of states
- $\Sigma$ is a (finite) alphabet
- $\delta: Q \times \Sigma \cup \{\varepsilon\} \to \mathcal{P}(Q)$ is the transition function
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states

**DFA (det. FA)**

- $\delta: Q \times \Sigma \to Q$ in det case:

Note: if $|Q|=n$ then $|\mathcal{P}(Q)|=2^n$
Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA and let $w = w_1w_2...w_n$ where each $w_i \in \Sigma$. Then $N$ accepts $w$ if

there exists a sequence of states $p_0, ..., p_n$ s.t.

1) same

2) $\delta(p_i, w_{i+1}) \in p_{i+1}$

3) same

$\text{in det. FA}$

if $M = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ is a DFA and $w = w_1...w_n$ then $M$ accepts $w$ if $\exists p_0, ..., p_n$ s.t.

1) $p_0 = q_0$

2) $\delta(p_i, w_{i+1}) = p_{i+1}$ $\forall i$

3) $p_n \in F_1$
Let $N=(Q, \Sigma, \delta, q_0, F)$ be an NFA and let $w=w_1w_2\ldots w_n$ where each $w_i \in \Sigma$. Then $N$ accepts $w$ if

$\exists \ u_1, \ldots, u_m \in \Sigma \cup \{\epsilon\} \ s.t. \ u_1u_2u_3\ldots u_m = w$ and

$\exists$ a sequence of states $p_0, \ldots, p_m \ s.t.$

1) $p_0 = q_0$

2) $\delta(p_i, u_{i+1}) \ni p_{i+1}$

3) $p_m \in F$
Thm 1.39: Every NFA has an equivalent DFA.

Proof idea:
- for starters, no $\varepsilon$-transitions in the NFA
- example:

Note: the # states in the DFA $\leq 2$ # states in the NFA

[Diagram of NFA and DFA]

Here have only "reachable" states
Thm 1.39: Every NFA has an equivalent DFA.

Proof idea:

- for starters, no ε-transitions in the NFA

- example: formal description of the construction:

  given an NFA $N = (Q, \Sigma, \delta, q_0, F)$, we describe an equivalent DFA $M = (Q_M, \Sigma, \delta_M, q_{0M}, F_M)$ as follows:

  $Q_M = \mathcal{P}(Q)$

  $q_{0M} = \{ q_0 \}$

  $F_M = \{ S \in Q \mid S \cap F \neq \emptyset \}$

  $\delta_M(S, \sigma) = \bigcup_{p \in S} \delta(p, \sigma)$ for all $S \subseteq Q$ and $\sigma \in \Sigma$
Thm 1.39: Every NFA has an equivalent DFA.

Proof idea, part 2 (getting rid of $\varepsilon$-transitions in the NFA):

- for $R \subseteq Q$ let:

$$E(R) = \{ q \in Q \mid q \text{ can be reached from } R \text{ by traveling along 0 or more } \varepsilon\text{-arrows } \}$$

Formally: Here NFA $N_1$, want to construct NFA $N_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$ where

- $Q_2 = Q_1$
- $q_{02} = q_{01}$
- $F_2 = F_1$

$\delta_2(p, \varepsilon) = E(\delta_1(p, \sigma)) + \text{other arrows}$

book: start states $E(\{q_{01}\})$

$$\delta_2(p, \varepsilon) = \bigcup_{r \in E(p)} E(\delta_1(r, \sigma))$$
Thm 1.45 (revisited): The class of regular languages is closed under the union operation.

Proof: restatement: if \( A, B \) are regular then \( A \cup B \) is reg.

(we proved this with DFA's)

Alternative proof: suppose \( N_A \) is a NFA for \( A \)

\( N_B \) is a NFA for \( B \)

want to construct a NFA \( N_{A \cup B} \) for \( A \cup B \)

\( N_{A \cup B} = (Q_{A \cup B}, \Sigma, \delta_{A \cup B}, q_{A \cup B}, F_{A \cup B}) \)

where

\( Q_{A \cup B} = Q_A \cup Q_B \cup \{q_{A \cup B}\} \)

\( q_{A \cup B} \subseteq q_{A \cup B} \)

\( F_{A \cup B} = F_A \cup F_B \)

\( \delta_{A \cup B}(q_{A \cup B}, \sigma) = \delta_A(q_{A \cup B}, \sigma) \)
Thm 1.47: The class of regular languages is closed under the concatenation operation.
Thm 1.49: The class of regular languages is closed under the star operation.

Pf: for a reg. lang. $A$ we want to prove $A^*$ is also regular

have $N_A$ want to construct $N_{A^*}$ for $A^*$

$$A^* = \bigcup_{k=0}^{\infty} A^k$$

$A^* = \{\varepsilon\}$