Which of these languages are regular?

- $B = \{ 0^n1^n \mid n \geq 0 \}$ not regular (see earlier slides)

- $C = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 0's and 1's} \}$

- $D = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 01's and 10's as substrings} \}$

$D$ is regular:
Which of these languages are regular?

- \( B = \{ 0^n1^n \mid n \geq 0 \} \)
- \( C = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 0's and 1's} \} \)
- \( D = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 01's and 10's as substrings} \} \)

Proof by closure properties:

We know \( B \) is not regular. We will show \( C \) is not regular.

By contradiction, assume \( C \) is regular.

\[ C \cap 0^*1^* = B \]

Assumed regular

Known to be regular

\( B \) is not regular. We know \( B \) is not regular.

Hence, \( C \) cannot be regular.
Pumping lemma for regular lang.

Suppose we have a DFA with \( p \) states.

Suppose there is a string of length \( \geq p \) that is accepted. Are there other strings that are accepted?

\[ s = s_1s_2s_3\ldots s_k , \quad k \geq p , \quad s_i \in \Sigma \]

- \( p \) states
- along the computation on \( s \), at least one state has to be revisited (visited twice)
- \( s = xy^2 \)
- \( \forall i \geq 0 : \ xy^i z \in L(M) \)
- \( y \neq \varepsilon \) \((|y| > 0)\)
**Thm 1.70 [pumping lemma]:**

Let $A$ be a regular language. Then there exists a number $p$ s.t. for every string $s \in A$ of length $\geq p$ there exist strings $x, y,$ and $z$ s.t.

0. $s = xyz,$
1. For each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0,$ and
3. $|xy| \leq p.$
**Pumping lemma for regular lang.**

**Example:** \( B = \{ 0^n1^n \mid n \geq 0 \} \)

Want to show \( B \) is not regular (by using the PL).

By contradiction, suppose \( B \) is regular.

Then, the PL holds for \( B \), i.e. \( \exists p \geq 1 \) s.t. \( \forall s \in B, |s| \geq p, \exists x,y,z \) satisfying (0)-(3)

**GAME:** Find \( s \in B, |s| \geq p \) but there do not exist \( x,y,z \) sat. (0)-(3)

\[
\begin{align*}
S = 01 & \in B \quad |s| \geq p \quad \checkmark \text{does not work.} \\
S = 0^{p-1}1^p & \in B \quad |s| = 2p \geq p \quad \checkmark
\end{align*}
\]

Suppose that \( x,y,z \) exist for \( s \) but then,

1. by 3) \( |xy| \leq p \) \( x,y \) contain only 0's
2. by 2) \( y \) contains at least one 0

Then, take \( i = 2 \): \( xy^iz = xyyz \) contains \( \geq p \) 0's.

\( \square \)
Example: $C = \{ w \mid w$ has equal number of 0's and 1's $\}$

suppose $C$ is regular. Then the PL holds for $C \rightarrow$ let $p$ be the PL number for $C$.

Consider $s = 0^p1^p \in C \checkmark \mid s \mid \geq p \checkmark$

from now on continue like on the previous slide \[\square\]
Example: \( F = \{ \text{ww} \mid w \in \{0,1\}^* \} \)

- **Strings in \( F \):** 0101, 1111, ε, 00, \( \lambda \), ...

Suppose \( F \) is regular. Then PL holds for \( F \). Let \( p \) be the PL number for \( F \).

Consider \( s = 01 \) bec. \( k \) >>>
- \( 0^p1^k \) \( \notin F \)
- \( 0^p0^p \) \( \notin F \)
- \( 0^p1 \) \( \notin F \)

But... take \( x = 0^p \), all conditions satisfied. \( \square \)

Need different \( k \)!

\[
0^p10^p1\]

\[
\begin{array}{c}
0^p \\
\hline
x = \frac{y = \frac{y^2 = 0^p}{z = 0^p}}{z = 0^p}
\end{array}
\]

- If \( k \), \( y \), \( z \) exist, then by 2), 3) \( y \) contains at least one 0 and contains only 0's.
- Then \( xyyz2 \in F \) \( \notin D \)
Example: $D = \{ 1^k \mid k \geq 0 \text{ is a square} \}$

Suppose $D$ is regular, then the PL holds for $D$. Let $p$ be the PL number for $D$.

Consider $s = 1^{3p} \in D$.

1. $1^p \in D$, $|s| \geq 3p$
2. $1^p \in D$, since $p$ might be non-square
3. $1^{3p} \in D$

Suppose $x, y, z$ exist, then by 2): $|y| > 0$, let $|y| = l \geq l$

By 3): $l \leq p$

(let's look at 1):

$$|xy^iz| = p^2 + (i-1) \cdot l$$

This grows linearly with $i$, i.e., cannot be all squares, but squares grow quadratically.

Consider $i = 2$:

$$xy^2 = xyyz^2$$

$$|xyz^2| = p^2 + 2 \leq p^2 + p < (p+1)^2$$

Therefore $xy^2 \notin D$.
Pumping lemma for regular lang.

Example: \( E = \{ 0^i1^j \mid i > j \} \)

Suppose that \( E \) is reg. Then the PL holds for \( E \). Let \( p \) be the PL number for \( E \).

Consider \( s = 0^21^p \in E \)

If \( |s| > p \), we have \( |x| = 0 \), \( |y| = 0 \), \( |z| = 0^p1^p \).

Conditions 0), 2), 3) are satisfied.

Let's look at \( s \):

\[ xy^lz = 0^{2p}1^{3p} \in E \]

i.e., we need different \( s \).

Suppose that \( x, y, z \) exist. By 3), 1):

- \( y \) contains only 0's and at least one 0

If \( i = 0 \):

- \( xy^lz = xz = 0^{ip}1^{ip} \in E \)
- \( k = k_1 \in E \)

[\( \square \)]