Determinism: computation always continues in a uniquely determined way.

Nondeterminism: have more (or none) choices

Example:

\[ \{ w \in \{0,1\}^* \mid w \text{ contains 001 or 0101 as a substring} \} \]
Nondeterminism

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Example:

\[ \{ w \in \{0,1\}^* \mid w \text{ contains 001 or 0101 as a substring} \} \]

Nondeterministic FA can also use \( \varepsilon \)-transitions:
Nondeterminism

Example:

\{ w \in \{0,1\}^* \mid w \text{ contains 1 in the third position from the end} \}

Does there exist a (deterministic) FA recognizing this language?

Yes (with 8 states!)

What if \( L = \{ w \in \{0,1\}^* \mid w \text{ contains 1 in the } \lfloor \frac{n}{2} \rfloor \text{ position from the end} \} \)

\( 2^n \) states for an NFA

\( 2^{\frac{n}{2}} \) for a DFA
Example:
\[ \{ w \in \{0\}^* \mid |w| \text{ is divisible by 3 or 4} \} \]
Nondeterminism

Formal definition:

A **nondeterministic finite automaton** (NFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- $Q$ is a finite set of states (nonempty)
- $\Sigma$ is a (finite) alphabet (nonempty)
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow \mathcal{P}(Q)$ is the transition function
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states

**we will use**
Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA and let $w = w_1w_2...w_n$ where each $w_i \in \Sigma$. Then $N$ accepts $w$ if $\exists r_0, r_1, ..., r_n \in Q$ s.t.

- $r_0 = q_0$
- $r_{i+1} \in \delta(r_i, w_{i+1}) \quad \forall i \in \{0, ..., n-1\}$
- $r_n \in F$
Thm 1.39: Every NFA has an equivalent DFA.

Proof idea:
- for starters, no \( \varepsilon \)-transitions in the NFA
- example:
Thm 1.39: Every NFA has an equivalent DFA.

Proof idea:
- for starters, no ε-transitions in the NFA
- construction:
  
  Let $N = (Q_N, \Sigma, \delta_N, q_N, F_N)$ be an NFA.
  We construct an equivalent DFA $M = (Q_M, \Sigma, \delta_M, q_M, F_M)$:

  Let:
  
  $$Q_M = \mathcal{P}(Q_N)$$

  $$q_M = \{ q_N \}$$

  $$\delta_M(S, \sigma) = \bigcup_{i=1}^{n} \delta_N(r_i, \sigma)$$

  where $S = \{ r_1, r_2, \ldots, r_n \}$

  or we could write
  
  $$\delta_M(S, \sigma) = \bigcup_{r \in S} \delta_N(r, \sigma)$$

  $$F_M = \{ s \in Q_M \mid s \cap F_N \neq \emptyset \}$$
Thm 1.39: Every NFA has an equivalent DFA.

Proof idea, part 2 (getting rid of $\varepsilon$-transitions in the NFA):
- for $R \subseteq Q$ let:

$$E(R) = \{ q \in Q \mid q \text{ can be reached from } R \text{ by traveling along 0 or more } \varepsilon\text{-arrows}\}$$

Example:

$$E(\{q_0,q_1\}) = \{q_0,q_1,q_2,9_4,9_5\}$$

$$E(\{q_2\}) = \{q_1,q_2,9_5\}$$
Thm 1.39: Every NFA has an equivalent DFA.

- for \( R \subseteq Q \) let:

\[
E(R) = \{ q \in Q \mid q \text{ can be reached from } R \text{ by traveling } \text{ along } 0 \text{ or more } \varepsilon \text{-arrows} \}
\]

let \( N = (Q_N, \Sigma, \delta_N, q_{N0}, F_N) \) be an NFA

we construct an equivalent DFA \( M = (Q_M, \Sigma, \delta_M, q_{M0}, F_M) \):

Let:

\[
Q_M = \mathcal{P}(Q_N)
\]

\[
q_M = E(\{ q_{M0} \})
\]

\[
\delta_M(s, \sigma) = E \left( \bigcup_{i=1}^{n} \delta_N(r_i, \sigma) \right) \quad \forall s \in Q_M, \quad \forall \sigma \in \Sigma
\]

where \( S = \{ r_1, r_2, \ldots, r_n \} \)

or we could write \( \delta_M(S, \sigma) = \bigcup_{r \in S} \delta_N(r, \sigma) \)

\[
F_M = \{ s \in Q_M \mid s \not\in F_N \}.
\]
Thm 1.45 (revisited): The class of regular languages is closed under the union operation.

Given two NFA $M_1, M_2$, we want to create a NFA $M$ such that

$$L(M) = L(M_1) \cup L(M_2)$$

Let:

$$Q = Q_1 \cup Q_2 \cup \{q_0\}$$

assuming: $Q_1 \cap Q_2 = \emptyset$

$$q_0 \notin Q_1, Q_2$$

$$F = F_1, \cup F_2$$

$$\delta(q, \sigma) = \begin{cases} 
\delta_1(q, \sigma) & q \in Q_1, \sigma \in \Sigma \\
\delta_2(q, \sigma) & q \in Q_2, \sigma \in \Sigma \\
\{q_1, q_2\} & q = q_0, \sigma = \varepsilon \\
\emptyset & q \neq q_0, \sigma \notin \Sigma 
\end{cases}$$
 Closure under regular operations

Thm 1.47: The class of regular languages is closed under the concatenation operation.

\[
\text{aabbabb}
\]

\[
\frac{\text{aabbabb}}{M_1, M_2}
\]

do the formal description yourself 😊
**Thm 1.49:** The class of regular languages is closed under the star operation.

Given a NFA $M_1$, we want to construct a NFA $M$ such that $L(M) = L(M_1)^*$.

Let $L(M_1) = \{a, bb\}$

Then $L(M_1)^* = \{\epsilon, a, bb, aa, aab, bba, bbb, \ldots\}$