The Halting Problem

\[ A_{TM} = \{ <M,w> \mid M \text{ is a TM that accepts string } w \} \]

- Turing-recognizable? \rightarrow YES

- Turing-decidable?

NO...

we'll see why

Sec. idea: simulate M on w, if M accepts, answer TRUE (trouble: figuring out when to say FALSE)
The Halting Problem

A bit about infinite sets and their sizes (diagonalization):

Def 4.12: Let $A,B$ be sets and let $f:A \rightarrow B$. We say that $f$ is

- **one-to-one** if $f(a) \neq f(b)$ for every $a \neq b$
- **onto** if for every $b \in B$ there exists $a \in A$ such that $f(a)=b$

If $f$ is one-to-one and onto, then $A,B$ are the **same size** and $f$ is called **correspondence**.

Example: $\mathcal{N} = \{1,2,3,4,5,...\}$ and $\{2,4,6,8,...\}$
The Halting Problem

A bit about infinite sets and their sizes (diagonalization):

Def 4.12: Let A, B be sets and let f: A → B. We say that f is
- **one-to-one** if f(a) ≠ f(b) for every a ≠ b
- **onto** if for every b ∈ B there exists a ∈ A such that f(a) = b

If f is one-to-one and onto, then A, B are the **same size** and f is called **correspondence**.

Example: \( \mathbb{N} = \{1, 2, 3, 4, 5, \ldots\} \) and \( \{2, 4, 6, 8, \ldots\} \)

Def 4.14: A set is **countable** if it is finite or has the same size as \( \mathbb{N} \).
Are \( \mathbb{Q} \) (rational numbers) and \( \mathbb{R} \) (real numbers) countable?
Cor 4.18: There is a language that is not Turing-recognizable.
The Halting Problem

Thm 4.11: $A_{TM}$ is not decidable.

Recall: $A_{TM} = \{ <M,w> | M \text{ is a TM that accepts string } w \}$

by contradiction, suppose decidable. Then there exist a function acceptance\((f, w)\):

outputs $\text{TRUE}$ iff $f(w) = \text{TRUE}$ (not a pseudocode).

Create another function:

weird\((f)\):

\[
\begin{align*}
\text{if } \text{acceptance}\((f, f)\) = \text{TRUE}, & \text{ return } \text{FALSE} \\
\text{else, return } \text{TRUE}
\end{align*}
\]

then, what happens if Weird\((\text{weird})\):

outputs $\text{FALSE}$ if $\text{acceptance}\((\text{weird, weird})\) = \text{TRUE}$ (i.e. weird\((\text{weird})\) = $\text{TRUE}$)

$\text{TRUE}$ if $\text{acceptance}\((\text{weird, weird})\) = \text{FALSE}$ (i.e. weird\((\text{weird})\) = $\text{FALSE}$ or infinite output)

A PARADOX! acceptance cannot exist $\Box$
The Halting Problem

Thm 4.22: A language $L$ is decidable iff $L$ is Turing-recognizable and $\overline{L}$ is Turing-recognizable (we say that $L$ is co-Turing-recognizable).

If: $\Rightarrow$ then immediately also $\overline{L}$-recognizable and co-$\overline{L}$-recognizable b/c. just swap accept & reject

$\Leftarrow$ TM $T_1$ for $L$ (accepts all strings in $L$, might do infinite computation for strings $\notin L$)

TM $T_2$ for $\overline{L}$ (if $L$)

idea for a TM-decider $T$: run $T_1$, $T_2$ in parallel (at the same time, e.g., using 2 tapes)

if $T_1$ accept, accept

$T_2$ accept, reject

Cor 4.23: $A_{TM}$ is not Turing-recognizable.

be $L$ if $A_{TM}$ is $\overline{L}$-recognizable. and we know that $A_{TM}$ is $\overline{L}$-recognizable, then by Thm 4.22, $A_{TM}$ would be decidable.