Nonregular languages

Which of these languages are regular?

- \( B = \{ 0^n1^n \mid n \geq 0 \} \) \( \text{not regular} \)
- \( C = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 0's and 1's} \} \) \( \text{not regular} \)
- \( D = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 01's and 10's as substrings} \} \) \( \text{regular} \)
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Proof by closure properties:

\text{[not in the book]}
Suppose we have a DFA with $p$ states.

Suppose there is a string of length $> p$ that is accepted. Are there other strings that are accepted?
Thm 1.70 [pumping lemma]:

Let \( A \) be a regular language. Then there exists a number \( p \) s.t. for every string \( s \in A \) of length \( \geq p \) there exist strings \( x, y, \) and \( z \) s.t.

0. \( s = xyz, \)

1. For each \( i \geq 0, \) \( xy^iz \in A, \)

2. \( |y| > 0, \) and \( i.e. y \neq \varepsilon \)

3. \( |xy| \leq p. \)
Pumping lemma for regular lang.

Example: \( B = \{ 0^n1^n \mid n \geq 0 \} \)

by contradiction, assume \( B \) is regular
then, the PL holds for \( B \). Let \( p \) be the PL number for \( B \).

it suffices to find one \( s \in B \) and \( |s| > p \) s.t. the 3 conditions cannot hold simultaneously.

consider \( s = 0^p1^p \)

and consider \( x, y, z \) satisfying conditions 0)-3)

then, by 3) \( |xy| \leq p \) and hence, \( xy \) contain only 0's

2) \( y \neq \epsilon \)

1) \( \forall i \geq 0 \ xy^i z \in B \) take \( i = 2 \) then \( xy^2z = xyyz \rightarrow \) will contain \( p + 1 \) 0's

\[ = p + |y| \geq p + 1 \]

hence \( xyyz \notin B \)
Example: $C = \{ w \mid w \text{ has equal number of 0's and 1's} \}$

by contradiction, assume $C$ regular, let $p$ be the PL number.

consider $s = 0011011 \not \in C$ we do not want this guy

$s = 0^p10^p0 \not \in C$ don't like

$s = 0011 \not \in C$ we do not want this guy since it might be shorter than $p$

$S = 0^p1^p$

argument same as for $B$

$S = \underbrace{0}_{i=2} \underbrace{1}_{i=0} \underbrace{0}_{i=1} \underbrace{1}_{i=1} \underbrace{0}_{i=1} \underbrace{1}_{i=1} \underbrace{0}_{i=1} \underbrace{1}_{i=1}$
Example: $F = \{ ww \mid w \in \{0,1\}^* \}$

by contradiction, assume $F$ is regular. Let $p$ be the PL number.

consider $s = 0^p1^p0^p1^p \in F \checkmark$

$|s| \geq p \checkmark$

we need to show that the conditions 0)-3) cannot hold simultaneously.

Suppose the conditions hold:

by 3) $|xy| \leq p$ i.e. $xy$ contain only 0's

2) $y \neq \varepsilon$

1) take $i = 2$: $xy^2z \in F$

$s = \overline{0101}$

$xy^2z = \overline{01011}$
Pumping lemma for regular lang.

Example: \( D = \{ 1^k | k \geq 0 \text{ is a square} \} \)

Suppose, by contradiction, that \( D \) is regular, then let \( p \) be the PL number.

Consider \( s = 1^{p^2} \in D \) \( \checkmark \)

\( |s| = p \) \( \checkmark \)

Suppose 0)-3) hold

by 3) \( |y| \leq p \)

2) \( |y| > 0 \)

1) Consider \( i = 2 \): \( xy^2z = xyyz = 1^{p^2 + |y|} \)

\[ p^2 \leq p^2 + |y| \leq p^2 + p \]

The next square is \( (p+1)^2 \)

\[
\frac{p^2 + 2p + 1}{p^2 + 2p + 1}
\]

\( xy^2z \in D \) \( \checkmark \)
Example: \( E = \{ 0^i1^j \mid i > j \} \)

Suppose, by contradiction, that \( E \) is regular. Let \( p \) be the PL number.

Consider
\[
    s = 0^{p+1}1^p \in E \quad \checkmark
\]
\[
    |s| > p \quad \checkmark
\]

Suppose 0)-3) hold:

by 3) \( |xy| \leq p \), i.e. \( xy \) contain only 0's

2) \( y \neq \varepsilon \)

1) \( i = 0 \quad xy_1^iz = xz \) - contains \( p \) 1's

\( xz \in E \quad \leq p \) 0's