Nondeterminism

Determinism: computation always continues in a uniquely determined way.

Nondeterminism: have more (or none) choices

Example:

\[ \{ w \in \{0,1\}^* \mid w \text{ contains 001 or 0101 as a substring} \} \]
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Nondeterministic FA can also use $\varepsilon$-transitions:
Nondeterminism

Example:
\{ w \in \{0,1\}^* \mid w \text{ contains 1 in the third position from the end} \}

Does there exist a (deterministic) FA recognizing this language?

Yes, but... we will need $2^k$ states.
Example:
\[ \{ w \in \{0\}^* \mid |w| \text{ is divisible by } 2 \text{ or } 3 \} \text{ with } \leq 8 \text{ states} \]
Nondeterminism

Formal definition:

A **nondeterministic finite automaton** (NFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- $Q$ is a finite set of states
- $\Sigma$ is a (finite) alphabet
- $\delta: Q \times (\Sigma \cup \{\varepsilon\}) \rightarrow \mathcal{P}(Q)$ is the transition function
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states
Let $N=(Q, \Sigma, \delta, q_0, F)$ be an NFA and let $w=w_1w_2...w_n$ where each $w_i \in \Sigma \epsilon$. Then $N$ accepts $w$ if there exists a sequence of states $r_0, r_1, ..., r_n$ such that:

1) $r_0 = q_0$

2) $r_{i+1} \in \delta(r_i, w_{i+1})$ for DFAs: $\delta(r_i, w_{i+1}) = r_{i+1}$

3) $r_n \in F$

For $w \in \Sigma^*$ we say that $N$ accepts $w$ if $\exists w_i \in \Sigma \epsilon$ such that:

$w = w_1w_2w_3...w_n$

and $\exists r_0, ..., r_n$ such that 1), 2), and 3) hold.
Thm 1.39: Every NFA has an equivalent DFA.

Proof idea:
- for starters, no ε-transitions in the NFA
- example:
Thm 1.39: Every NFA has an equivalent DFA.

Proof idea, part 2 (getting rid of \( \varepsilon \)-transitions in the NFA):
- for \( R \subseteq Q \) let:

\[
E(R) = \{ q \in Q \mid q \text{ can be reached from } R \text{ by traveling along } 0 \text{ or more } \varepsilon\text{-arrows} \}
\]

\[
\begin{align*}
E(\{q_5\}) &= \{q_5\} \\
E(\{q_0\}) &= \{q_0, q_1, q_2, q_3, q_5\} \\
E(\{q_2, q_4\}) &= \{q_2, q_4, q_3, q_5\}
\end{align*}
\]

s.t. \( L(N) = L(M) \)

Let:
- \( Q_M = \mathcal{P}(Q_N) \)
- \( q_{0M} = E(q_{0N}) \)
- \( F_M = \{ S \in Q_M \mid S \cap F_N \neq \emptyset \} \)
- \( \delta_M(S, \sigma) = E(\bigcup_{S' \in S} \delta_N(S', \sigma)) \forall S \in Q_M, \forall \sigma \in \Sigma \)
Thm 1.45 (revisited): The class of regular languages is closed under the union operation.

Proof with NFAs:

Given two NFAs $N_1, N_2$:

Let $N_1 = (Q_1, \Sigma_1, \delta_1, q_{10}, F_1)$ and $N_2 = (Q_2, \Sigma_2, \delta_2, q_{20}, F_2)$.

We construct an NFA $N$: $N = (Q, \Sigma, \delta, q_{0}, F)$

where

$Q = Q_1 \cup Q_2 \cup \{q_{\text{new}}\}$

and $q_{\text{new}} \notin Q_1 \cup Q_2$.

$F = F_1 \cup F_2$.

$\Sigma = \Sigma_1 \cup \Sigma_2$.

The transition function $\delta(q, \sigma)$ is defined as:

$\delta(q, \sigma) = \begin{cases} \delta_1(q, \sigma) & \text{if } q \in Q_1, \sigma \in \Sigma_1 \cup \{\epsilon\} \\ \delta_2(q, \sigma) & \text{if } q \in Q_2, \sigma \in \Sigma_2 \cup \{\epsilon\} \\ \{q_{\text{new}}\} & \text{if } q = q_{\text{new}}, \sigma = \epsilon \\ \emptyset & \text{otherwise} \end{cases}$

Assume (w.l.o.g.) $Q_1 \cap Q_2 = \emptyset$. 

Additional notes on the transition function and the construction of the NFA.
Thm 1.47: The class of regular languages is closed under the concatenation operation.

Proof with NFAs

"by boxes"

Given by NFAs $N_1, N_2$ for any two regular languages, we want to show that their concatenation is also regular. Construct an NFA $N$ s.t. $L(N) = L(N_1)L(N_2)$.
Thm 1.49: The class of regular languages is closed under the star operation.

Proof by "boxes"