Which of these languages are regular?

- \( B = \{ 0^n1^n \mid n \geq 0 \} \times \)
- \( C = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 0's and 1's} \} \times \)
- \( D = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 01's and 10's as substrings} \} \textbf{ regular: } (1^*01^*)^*1^*u(0^*1^*)^*0^* \)

\( B \) is not regular:

Suppose, by contradiction, that \( B \) is regular and let \( M \) be a DFA for \( B \). Consider strings of the form \( 0^k \) for a countably infinite number of such strings, but a finite number of states, thus:

- There must be \( 0^k1^l \) and \( 0^k \) ends in the same state as \( 0^l \).
- \( 0^k1^l \) should be accepted.
- \( 0^k1^l \) shouldn't be accepted but they go together (contradiction).
- So, \( M \) cannot exist for \( B \).
Nonregular languages

Which of these languages are regular?

- \( B = \{ 0^n1^n \mid n \geq 0 \} \)
- \( C = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 0's and 1's} \} \)
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Proof by closure properties:

We'll show that \( C \) is not regular, assuming that \( B \) is not regular. If \( C \) is regular, this must be regular because regular languages are closed under \( \cap \).

\[
B = C \cap 0^*1^* \quad \text{regular}
\]

\( \uparrow \)

by contradiction, assume \( C \) is regular

\( \therefore \) contradiction, \( B \) is known to be nonregular
Pumping lemma for regular lang.

Suppose we have a DFA with $p$ states.

Suppose there is a string of length $> p$ that is accepted. Are there other strings that are accepted?

Since $xyz$ is accepted and $y$ starts and finishes in the same state (there always is such a $y$ because we have only $p$ states so a state must be repeated on the computation path of $xyz$ since $|xyz| > p$), then $xyyz$ is also accepted; also $xz$ is accepted (skip the loop).

Thus $xy^iz$ is accepted for $i \geq 0$.
Thm 1.70 [pumping lemma]:

Let $A$ be a regular language. Then there exists a number $p$ s.t. for every string $s \in A$ of length $\geq p$ there exist strings $x, y,$ and $z$ s.t.

0. $s = xyz,$
1. For each $i \geq 0,$ $xy^iz \in A,$
2. $|y| > 0,$ and
discussed on the previous slide
3. $|xy| \leq p.$

\[ \text{Reading } |xy| \text{ symbols means that we go through } |xy|+1 \text{ states} \]
\[ \text{Thus, going through the first } p \text{ symbols means that we visit } p+1 \text{ states, thus there must be a repeating state defining } y \]
\[ \Rightarrow |xy| \leq p \]
Example: \( B = \{ 0^n1^n \mid n \geq 0 \} \)

nonregular, we'll show this by contradiction:

Suppose regular, then the PL holds; let \( p \) be the PL constant for \( B \)

the PL needs to hold for \( s = 0^p1^p \) verify: \( s \in B \) √

\(|s| \geq p \) √

\[ S = \begin{array}{cccc}
& & & P \\
& 0 & \ldots & 0 & 1 & \ldots & 1 & \text{there should be a split of } s \text{ into } x, y, z \\
\end{array} \]

\[ x \quad y \quad z \]

Case analysis:

if \( y \) contains both 0's and 1's, then \( xyz^2 = 0^{p-1}10^{p-1}1 \) \& \( B \)

if \( y \) contains only 0's, then \( xyz^2 \) contains more zeros than ones \& \( B \)

if \( y \) contains only 1's, same argument as for 0's \& \( B \)

Thus, no such \( x, y, z \) for our \( s \) \( \Rightarrow \) PL does not hold \( \Rightarrow \) \( B \) is not regular
Pumping lemma for regular lang.

Example: \( C = \{ w \mid w \text{ has equal number of 0's and 1's} \} \)

Suppose \( C \) is regular, then the PL holds, let \( p \) be the PL constant.

Let \( s = 0^p1^p \in C \), \( |s| = 2p \geq p \)

Same argument as before works.

\[ s = \underbrace{0 \ldots 0}_{P} \underbrace{1 \ldots 1}_{P} \]

By 3) \( xy \) contains only zeros.

Let \( i = 2 \), then \( xy^2 \) contains more zeros than ones \( \notin C \)

\( \sqrt{ } \) contradiction, with \( C \) regular.
Example: \( F = \{ ww \mid w \in \{0,1\}^* \} \)

Suppose \( F \) is regular. Then the PL holds, let \( p \) be the PL constant.

Goal: Find \( s \) that results in a contradiction.

Let \( s = 01^p01^p \in F \), \( |s| = 2p+2 \geq p \)

Do there exist \( x, y, z \)?

Case analysis:
- \( y \) either contains the first 0 and some of the first set of ones
  - \( i = 2 \): \( xy^kz = 01^m01^01^p \) where \( m < p \) \& \( F \)
- or none
- \( y \) contains some of the first set of ones
  - Then \( i = 2 \): \( xy^kz = 01^m01^p \) where \( m \geq p+1 \) \& \( F \)

\[ s = \begin{array}{c}
01 \ldots 1 \\
|y| \\
\ldots \\
01 \ldots 1 \\
z
\end{array} \]
Example: \( D = \{ 1^k \mid k \geq 0 \text{ is a square} \} \)

Suppose \( D \) is regular, then the PL holds, let \( p \) be the PL constant.

Let \( s = 1^p \).

Do there exist \( x, y, z \)?

If yes:\n\[ x = 1^k, \quad y = 1^l, \quad z = 1^{p^2-k-l} \]

By 2): \( l > 0 \)

3) \( k+l \leq p \)

By 1): let \( i = 2 \): \( xyyz = 1^{p^2+l} \)

Cannot be a square length since the next square after \( p^2 \) is \( p^2+2p+1 \)

But \( l \leq p \)

\( \in D \)
Pumping lemma for regular lang.

Example: \( E = \{ 0^k1^j | k > j \} \)

not regular: by contradiction, suppose regular, let \( p \) be the PL constant

| Let \( S = 0^{p+1}1^p \in E \) ✓ |
| \( |S| = 2p + 1 \geq p \) ✓ |

\[
S = \begin{array}{ccc}
0 & \cdots & 0 \\
1 & \cdots & 1 \\
\end{array}
\]

if there are \( x, y, z \) satisfying 0) 2) 3):

by 0) \( S = xy^2 \)

2) \( y \neq \varepsilon \)

3) \( xy \) contains only 0's

1) if \( i \geq 1 \), then \( xy^i z \in E \) ✓ not helpful for the contradiction

let \( i = 0 \), then \( xy^i z = xz \) contains \( \leq p \) zeros \( = p \) ones \& \( E \) ✓