Nonregular languages

Which of these languages are regular?

- $B = \{ 0^n1^n \mid n \geq 0 \}$ not regular, we'll see why
- $C = \{ w \in \{0,1\}^* \mid w \text{ has equal number of } 0\text{'s and } 1\text{'s } \}$
- $D = \{ w \in \{0,1\}^* \mid w \text{ has equal number of } 01\text{'s and } 10\text{'s as substrings } \}$
Nonregular languages

Which of these languages are regular?

- \( B = \{ 0^n1^n \mid n \geq 0 \} \)
- \( C = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 0's and 1's} \} \)
- \( D = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 01's and 10's as substrings} \} \)

Proof by closure properties:

We showed that regular lang. are closed under: \( \cup, \cdot, *, \cap, \text{ complement} (\neg) \)
we will show: if \( B \) not regular, then \( C \) not regular
i.e., contrapositive: if \( C \) is regular, then \( B \) is regular

IDEA: using closure properties, we'll construct \( B \) from \( C \)

\[ B = C \cap a^*b^* \]

assumed reg.  
intersection of (supposedly) reg. lang. 
\( \Rightarrow \) a regular lang. \( (B) \)
Suppose we have a DFA with p states.

Suppose there is a string of length > p that is accepted. Are there other strings that are accepted?

The strings that are accepted are:

- $u^*v^*w$

Notice that if a string is > p, we have a repetition.

Thus $|uv| \leq p$

$X = x_1x_2...x_k$

where $k \geq p$

$X$ during its computation goes through a sequence of $k+1$ states, i.e. some states must repeat.
Thm 1.70 [pumping lemma]:

Let $A$ be a regular language. Then there exists a number $p$ s.t. for every string $s \in A$ of length $\geq p$ there exist strings $x, y,$ and $z$ s.t.

0. $s = xyz,$
1. For each $i \geq 0$, $xy^iz \in A,$
2. $|y| > 0$, and
3. $|xy| \leq p.$

Note:

previous page, we started with $x$ and chopped it into $u,v,w$

Here we are chopping $s$ into $x,y,z$.

We'll be using the PL to show that $A$ is not regular.

Recipe: assume $A$ is regular. Then the PL holds for $A$. Thus, if we find $s$, longer than $p$ that does not satisfy (0)-(3) $\Rightarrow$ PL does not hold for $A$ $\Rightarrow$ it is not regular.


**Example:** \( B = \{ 0^n1^n \mid n \geq 0 \} \)

we'll show \( B \) is not regular

by contradiction, assume \( B \) is regular. thus the PL holds.

I.e. there exists \( p \), the pumping lemma constant (for \( B \)).

Idea: show that there is a string \( s \in B \) of length \( 2p \) s.t. \( s \) does not satisfy 0)-3).

let's take \( s = 0^p1^p \in B \). √

is \( s \) longer than \( p \)? √

let's try to make 0)-3) hold:

by 3) we know that \( xy \) contain only 0's

2) \( y \) contains at least one zero

1) \( xy^iz \) for \( i=2 \)

\[ s = 0^p1^p \]

\[ s = xy^i\tilde{z} \]

\( x \) has exactly \( p \) ones \( \tilde{z} \) has more than \( p \) zeros

therefore \( xyyz \notin B \)

then 1) does not hold

\( \Rightarrow \) the PL does not hold for \( B \)

\( \Rightarrow \) \( B \) is not regular
Pumping lemma for regular lang.

Example: \( C = \{ w \mid w \text{ has equal number of 0's and 1's} \} \)

by contradiction, assume \( C \) is regular
then the PL holds for \( C \) : there is the pl. constant \( p \) for \( C \)
therefore every string in \( C \) of length \( \geq p \) must satisfy 0)-3)

IDEA: produce one string \( s \) s.t. \( s \in C \) and \( |s| \geq p \)
for which 0)-3) cannot hold

Can we take:
\( s = \text{0011} \) \( \not\in \text{not in } C, \not\text{OK} \)
\( s = \text{0011} \in C, \text{ but not longer than } p \) (we do not know if it's longer, depends on \( p \), but we only know that \( p \) exists, not the actual value)

Let's consider:
\( s = \text{0}^p1^p \in C \checkmark \)
\( |s| = 2p \geq p \checkmark \)
we'll see if 0)-3) can hold
the same argument as previous slide \( \square \)
Pumping lemma for regular lang.

Example: \( F = \{ \text{ww} \mid w \in \{0,1\}^* \} \)

by contradiction, assume \( F \) regular and let \( p \) be the PL constant,
we'll find \( s \in F \) of length \( \geq p \) s.t. (0)-(3) cannot hold \( \forall \) contradiction

let's take \( s = (01)^p(01)^p \)

\[ S = 010101\ldots01 \]

Then \( x = 01 \)
\( y = 01 \)
\( z = (01)^{2p-2} \)

satisfy (1)-(3) \( \rightarrow \) this \( s \) does not do the trick

another attempt:
\( S = 0^p1^p0^p1^p \)

\[ S = 0^p|1^p|0^p|1^p \]

let's consider an \( xy^iz \) split satisfying (2) and (3):

by (3) \( xy \) contains only 0's from the initial 0-segment

by (2) \( y \) contains \( \geq 1 \) zero

by (1) any \( i \geq 0 \) \( xy^iz \in F \) should hold

let's \( i = 2 \) \( xy^2z \) \( \notin \) the first block of zeros has length \( > p \)

followed by \( 1^p0^p1^p \) therefore \( xyz2 \notin F \)

\( \square \)
Example: $D = \{ 1^k \mid k \geq 0 \text{ is a square} \}$  

by contradiction  
suppose $D$ is regular. let $p$ be the PL constant.  
we'll find $s \in D$ of length $\geq p$ s.t. (1-3) do not hold.  

Take $s = 1^p \in D$, length $|s| = p^2 \geq p$  

Suppose $s = xyz$ by  
\begin{align*}  
1) & \forall i \geq 0: \quad xy^iz \text{ must be in } D \\
2) & y \text{ contains at least one one} \\
3) & |xy| \leq p \\
\end{align*}  

$m \leq p$  
$m \geq 1$  

Therefore $xy^2z$ cannot be of a square length.  
thus $s \not\in D$  

Can we choose $i$ s.t. $xy^iz$ is not a square?  
let's consider $i = 3$  

$|xy^iz| = p^2 + m(i-1)$  

$p^2 < |xy^iz| \leq p^2 + 2p$  

the next larger square after $p^2$: $(p+1)^2 = p^2 + 2p + 1$  

but larger than $p^2$  

$|xy^iz| = p^2 + 2m \leq p^2 + 2p$  

$
Pumping lemma for regular lang.

**Example:** \( E = \{ 0^i 1^j | i > j \} \)

Suppose \( E \) regular, let \( p \) be the PL constant.

Let's take
\[
S = 0^{p+1} 1^p \in E \quad \text{and} \quad |S| = 2p + 1 \geq p
\]

Let's consider
\[
S = xy^2z
\]

By 3) \( xy \) contain only 0's

2) \( y \) contains \( \geq 1 \) zero

1) \( \forall i \geq 0 \): \( xy^iz \) should be in \( E \)

Let's find \( i \) s.t.
\[
xy^iz \in E \quad \text{then we'll be done}
\]

Let's take \( i = 0 \): (no other \( i \) works)

\[
xy^0z = xz \quad \text{# of ones is} \quad p \quad \text{# of zeros is} \quad \leq p + 1 - 1 = p
\]

Thus \( xy^0z \in E \) \( \subseteq \) original

**TRY:**
\[
\{ 1^k | k \geq 0 \text{ is a prime} \}
\]

\[
S = \overline{0^{p+1} 1^p} = \frac{\overline{0^{p+1}}}{x} \overline{1^p} = \frac{y}{\overline{y}} \overline{z}
\]