Nonregular languages

Which of these languages are regular?

- $B = \{ 0^n1^n \mid n \geq 0 \}$ not regular
- $C = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 0's and 1's} \}$ not regular
- $D = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 01's and 10's as substrings} \}$ is regular. E.g., reg. expr: $0(01)^*0 \cup 1(01)^*1 \cup \varepsilon \cup 01$
Nonregular languages

Which of these languages are regular?

- $B = \{ O^n1^n \mid n \geq 0 \}$
- $C = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 0’s and 1’s} \}$
- $D = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 01’s and 10’s as substrings} \}$

Proof by closure properties:

Assume $B$ is not regular.
We will show that $C$ is not regular either.
By contradiction, suppose $C$ is regular. Consider $C \cap 0^*1^* = B$.

Regular because regular languages are closed under intersection.
Assume regular, know regular, contradiction.

Hence $C$ is not regular.
Pumping lemma for regular lang.

Suppose we have a DFA with $p$ states.

Suppose there is a string of length $> p$ that is accepted. Are there other strings that are accepted?

3) $|xy| \leq p$
   
   Bec. within the first $p$ symbols we go through $p+1$ states and hence have a repeated state

\[ S = xy^2 \]

Also accepted:
\[ x^2 \]
\[ xy^nz \]

1) $xy^2z$ is accepted for $n \geq 0$

2) $y \neq \varepsilon$ or $|y| > 0$
Thm 1.70 [pumping lemma]:

Let $A$ be a regular language. Then there exists a number $p$ s.t. for every string $s \in A$ of length $\geq p$ there exist strings $x, y,$ and $z$ s.t.

0. $s = xyz,$
1. For each $i \geq 0,$ $xy^iz \in A,$
2. $|y| > 0,$ and
3. $|xy| \leq p.$
Example: \( B = \{ 0^n1^n \mid n \geq 0 \} \)

Proof: That \( B \) is not regular.

By contradiction, assume that \( B \) is regular and let \( p \) be the PL number for \( B \).

Goal: find one \( s \in B \), \( |s| \geq p \) for which the PL does not hold (there is a problem w. the 3 conditions).

Consider \( s = 0^p1^p \in B \), \( 0^p \times \) not ok, \( 1^p \times \) ok, \( s \in B \) but \( |s| \) could be \( \geq p \). \( s = \frac{0}{0} \frac{1}{1} \frac{p}{p} \frac{p}{p} \frac{1}{1} \frac{2}{2} \)

We will find a problem w. condition (1) (the other conditions are easy to satisfy)

pretend that all conditions hold:

by 3') \( xy \) happen within the 0 block, i.e. \( y \) contains only 0's

2'). \( y \neq \epsilon \)

1') consider \( i = 2 \) : \( xy^i2 = xyy2 \) - contains \( \geq p \) 1's

\( \geq p \) 0's (and \( \leq 2p \))

and hence \( B \) is not regular.
Pumping lemma for regular lang.

Example: \( C = \{ w \mid w \text{ has equal number of 0's and 1's} \} \)

Proof that \( C \) is not regular.

By contradiction, assume \( C \) is regular and let \( p \) be the PL number for \( C \).

Consider \( s = \overline{011001} \in C \) \( \forall \) not necessarily long enough

\[
S = O^p 1^p
\]

Suppose \( x, y, z \) satisfy:

1) \( s = xy^2 \)
2) \( y \) contains at least one 0
3) \( |xy| \leq p \), i.e. \( y \) contains only 0's

Consider \( i = 2 \), then \( xy^2z = xyyz \) contains \( \geq p \) 1's

\( C \) \( \notin \) regular.
Pumping lemma for regular lang.

Example: $F = \{ ww \mid w \in \{0,1\}^* \}$

Pf that $F$ is not regular.

By contradiction, suppose $F$ regular, let $p$ be the PL number.

Consider $S = 00$

- $0001$: too short
- $(01)^p$: too short & $F$
- $0^p1$: $F \not\subset F$
- $0^p10^p1$: too short

Let $S = 0^p10^p1$

Suppose $x,y,z$ satisfy (0)-(3): 0) $S = xyz$
1) consider $i = 2$: $xy^iz = xyyz = 0^k10^p1 \in F$
2) $y \neq \varepsilon$ contains at least one 0
3) $|xy| \leq p$ $y$ contains only 0's

If we chose $S = (01)^p$

then $3x,y,z$ satisfying the condition,

- $x = \varepsilon$
- $y = 0101$
- $z = (01)^p \cdot 2$

$\forall i \geq 0$ $xy^iz \in F$

Not a good choice for $S$
Example: \( D = \{ 1^k \mid k \geq 0 \text{ is a square} \} \)

Proof that \( D \) is not regular:

By contradiction, assume \( D \) is regular. Let \( p \) be the PL number for \( D \).

Consider \( s = 1^{p^2} \in D \) \( \forall \) \( |s| = p^2 \geq p \)

Suppose \( x, y, z \) satisfy 0)-(3):

by 2) : \( y \) contains at least one 1 \( \quad |y| \geq 1 \)

3) : \( |xy| \leq p \) \( \quad |y| \leq p \)

1) : \( xy^iz = xyy \ldots yz \) \( \underbrace{y}_{i \text{ times}} \)

consider \( i = 2 \) : \( xy^2z = xyyz = 1^{p^2+|y|} \)

\( p^2 + 1 \leq xyyz \leq p^2 + p \)

The next square after \( p^2 \) is \( (p+1)^2 = p^2 + 2p + 1 \)

hence no square \( (p^2+1, p^2+p] \):

\( xyyz \notin D \) \( \square \)
**Example:** \( E = \{ 0^i 1^j \mid i > j \} \)

We will show that \( E \) is not regular.

By contradiction, suppose \( E \) is regular. Then let \( p \) be the PL number for \( E \).

Consider \( s = 0^{p+1} 1^p \in E \) \( \Rightarrow |s| = 2p + 1 \geq p \).

Suppose \( x,y,z \) satisfy 0)-(3):

by 3): \( |xy| \leq p \) \( \Rightarrow y \) contains only 0's

2): \( y \) contains at least one 0

1): consider \( i=2 \): \( xy^2z = xyyz = 0^p 1^p \in E \)

\( s = \begin{array}{c}
0 \quad 1 \\
p+1 \\
p \\
\end{array} \)

This i (i.e.) does not produce contradiction.

Consider \( i=0 \): \( xy^0z = xz = 0^{p+1} 1^p \in E \)

i.e. \( xz \) contains \( p+1-1 \) zeros

\( = p \) ones

\( \Rightarrow xz \in E \).