Reductions: if we can reduce (transform) problem A into a problem B, then solving problem B gives solution to problem A.

Example: $\text{HALT}_{TM} = \{ <M,w> | M \text{ is a TM that halts on } w \}$

we will show that $\text{HALT}_{TM}$ is undecidable

by contradiction, assume $\text{HALT}_{TM}$ is decidable and let us have a TM decider for $\text{HALT}_{TM}$:

Thm 5.1: $\text{HALT}_{TM}$ is undecidable.

Note: $\text{HALT}_{TM}$ is the halting problem, $A_{TM}$ is the acceptance problem.
Thm 5.1: $E_{TM}$ is undecidable, where

$$E_{TM} = \{ <M> \mid M \text{ is a TM and } L(M) = \emptyset \}$$

by contradiction, suppose $E_{TM}$ is decidable:

- Create $M_2$ as follows:
  1. erase the input
  2. write $w$ on the tape
  3. run $M$ on the contents of the tape ($w$)

Notice that $L(M_2) = \begin{cases} \Sigma^* & \text{if } M \text{ accepts } w \\ \emptyset & \text{otherwise} \end{cases}$

Thus, $A_{TM}$ is decidable if $E_{TM}$ is decidable

$\Rightarrow E_{TM}$ is undecidable
Thm 5.3: REGULAR$_{TM}$ is undecidable, where

REGULAR$_{TM}$ = \{ <M> | M is a TM and L(M) is regular \}
Thm 5.4: $EQ_{TM}$ is undecidable, where

$$EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TM's and } L(M_1) = L(M_2) \}$$
Thm 5.4: $\text{ALL}_{\text{CFG}}$ is undecidable, where

$$\text{ALL}_{\text{CFG}} = \{ \langle G \rangle \mid G \text{ is a CFG and } L(G) = \Sigma^* \}$$
Rice’s Thm [Problem 5.28]:

Let $p$ be a language property. If $p$ holds for some but not all languages, then the following language is undecidable:

$$R = \{ <M> \mid M \text{ is a TM and } L(M) \text{ satisfies } p \}$$