Nonregular languages

Which of these languages are regular?

- \( B = \{ 0^n1^n \mid n \geq 0 \} \) – proved non-regular by Myhill-Nerode

- \( C = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 0's and 1's} \} \) – not regular

- \( D = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 01's and 10's as substrings} \} \)

\( C \) is not regular by Myhill-Nerode bec.

\[ \forall i, j : \quad O^i \neq_c O^j \quad \text{bec. for } z = 1^i : \]

\( 0^i 2 = 0^i 1^i \in C \)

\( 0^i 2 = 0^i 1^i \notin C \)

Thus, \([0^i]_c\) form co-number of equivalence classes

\[ \Rightarrow \quad \text{non-regular} \]
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Proof by closure properties:

\( C \) is not regular:

Suppose, by contradiction, that \( C \) is regular
then, notice:

\[ B = C \cap 0^*1^* \]

\[ \uparrow \text{regular lang.} \]
\[ \text{assumed regular} \]

\( C \cap 0^*1^* \) must be regular
by contradiction, since \( B \) is known to be not regular
Pumping lemma for regular lang.

Suppose we have a DFA with $p$ states.

Suppose there is a string of length $> p$ that is accepted. Are there other strings that are accepted?

Example:

- $i=2$: $xyyz$ accepted
- $i=3$: $xxyyz$ accepted
- ...
- $i=1$: $xyz$ accepted
- $i=0$: $xz$ accepted

Notice:

- $y \neq \varepsilon$ bec. the transitions on the cycle are labeled by symbols, not $\varepsilon$ in a DFA
- the first cycle occurs within the first $p+1$ states (p symbols) \( \Rightarrow |xy| \leq p \)

if $|s| \geq p$ then along the computation on $s$ we visit $\geq p+1$ states \( \Rightarrow \) at least one state is repeated

the repeated state (the first such state)

Example:

\[ S = x y y z \]

$xyiz$ accepted for $i \geq 0$
Thm 1.70 [pumping lemma]:

Let $A$ be a regular language. Then there exists a number $p$ s.t. for every string $s \in A$ of length $\geq p$ there exist strings $x$, $y$, and $z$ s.t.

0. $s = xyz$,

1. For each $i \geq 0$, $xy^iz \in A$,

2. $|y| > 0$, and

3. $|xy| \leq p$.

We will use the PL to show that a language is not regular:

Outline: Suppose $L$ is regular, then the PL holds for $L$, let $p$ be the PL number for $L$.

$\Rightarrow$ it suffices to find a string $s \in L$, $|s| \geq p$ s.t. we cannot decompose $s$ into $xyz$ for which 1-3 hold.

$\Rightarrow$ contradiction w PL $\Rightarrow L$ is not regular.
Example: \( B = \{ 0^n1^n \mid n \geq 0 \} \)

by contradiction, assume \( B \) is regular, then the PL holds for \( B \), let \( p \) be the PL number for \( B \n
\text{consider } s = 0^p1^p \in B \checkmark \quad |s| = 2p = p \checkmark 

by the PL, should be able to decompose \( s \) into \( x,y,z \) satisfying all conditions 1)-3)

(*) \quad \begin{cases} 
\text{if } x,y,z \text{ satisfy 2): } & y \neq E \\
3): & (xy)_i = p \\
& xy \text{ contain only zeros} \\
1): \text{ for all } i \geq 2: & xy^iz \in B \\
& \text{let's try } i = 2: & xy^2z \in B \quad \text{has } p \text{ ones} \\
& \text{has } >p \text{ zeros bec. we have extra zeros from the extra } y \\
& \text{by contradiction, } & B \text{ cannot be regular} \\
\end{cases}

Pumping lemma for regular lang.

Example: \( C = \{ w \mid w \text{ has equal number of 0's and 1's} \} \)

not regular, we'll show by the PL

by contradiction, assume \( C \) is regular, then the PL holds, let \( p \) be the PL number for \( C \)

consider \( s = 0^p1^p \in C \checkmark \quad |s| = 2p \geq p \checkmark \)

follow the same argument as before (see (x))
Example: \( F = \{ \, ww \mid w \in \{0,1\}^* \, \} \)

Some strings in \( F \): \( 00, 11, 0101 \)

\( F \) is not regular:

Suppose, by contradiction, \( F \) is regular, let \( p \) be the PL number for \( F \).

Consider \( s = 0^p1^p0^p1^p \in F \), \( |s| = 4p \geq p \).

Suppose we would have \( x, y, z \) satisfying 1) \begin{itemize}
  \item by 1) \( s = xyz \)
  \item 2) \( y \neq \varepsilon \) at least one zero
  \item 3) \( |xy| \leq p \) only zeros
\end{itemize}

4) \( \forall i \geq 0: \ xy^iz \) should be in \( F \)

\( i=2: \ xy^{i}z = 0^r1^p0^p1^p \in F \), where \( r = p + |y| \geq p + 1 \).
Pumping lemma for regular lang.

Example: \( D = \{ 1^k | k \geq 0 \text{ is a square} \} \)

\( D \) contains: \( \epsilon, 1, 1111, 1^9, 1^{16}, \ldots \)

Suppose \( D \) is regular (by contradiction), let \( p \) be the PL number for \( D \)

Consider \( s = 1^{p^2} \in D \) \( \land |s| = p^2 \geq p \)

Suppose \( x, y, z \) satisfy 0)-3):

by 0) \( s = xyz \)
2) \( y \neq \epsilon \) \( \rightarrow y \) contains at least one 1
3) \( |xy| \leq p \)
4) \( \forall i \geq 0 : xy^iz \in F \)

Consider \( i = 0 : xy^iz = xz = 1^{p^2 - |y|} \notin F \downarrow \)

The nearest smaller square to \( p^2 \) is \( (p-1)^2 = p^2 - 2p + 1 \)

But \( p^2 - |y| \leq p^2 - 1 \leq p^2 - p \) thus cannot be a square since \( p^2 - p > p^2 - 2p + 1 \)
Pumping lemma for regular lang.

Example: $E = \{ 0^k1^j \mid k > j \}$

we'll show that $E$ is not regular.

by contradiction, assume $E$ is regular. then the PL must hold for $E$, let $p$ be the PL number.

goal: to find a string $s \in E$ with $|s| \geq 2p$ s.t.: for no $x, y, z$ all conditions 0)-3) can hold simultaneously.

let $s = \begin{cases} \sigma_1^p & \text{not ok, its might be shorter than } p \\ \sigma_1^{p+1} & \text{not ok, not in } E \\ 0^{p+1} & \text{ok, in } E, \ |s|=2p+1 \geq 2p \text{ + see the argument below} \\ 0^p & \text{in } E, \ \text{of length } p \text{ but cannot be used to show the contradiction} \end{cases}$

let $s = 0^{p+1}1^p$

$\sigma_1^{p+1} = \sigma_1^{p} \sigma_1 \sigma_1^{p}$

imagine that there would be $x, y, z$ satisfying 0)-3):

then by
0) $s = xy^2z$
1) $|y| \geq 1$
3) $|xy| \leq p \ \text{xy contain only } 0'$s
1) $i \geq 0: \ xy^iz \in E$

let's look at $i = 2$: $xy^2z \in E$ bec. #0's increased So

$i = 0: \ \text{does not exhibit the problem (show nonexistence of } x, y, z)$

$\#0' \text{'s } > \#1'$s