The Halting Problem

\[ A_{TM} = \{ <M,w> \mid M \text{ is a TM that accepts string } w \} \]

- Turing-recognizable? \textbf{Yes} \hspace{1cm} \text{bec:} \hspace{1cm} \text{"Alg": 1. simulate } M \text{ on } w \\

  2. if } M \text{ accepts, accept (return true)} \\
  3. if } M \text{ rejects, reject (return false)} \\
  \text{troubles: infinite computation possible} \\
  \text{i.e. not a decider}

- Turing-decidable? \textbf{NO}
The Halting Problem

Thm 4.11: $A_{TM}$ is not decidable.

Recall: $A_{TM} = \{ <M,w> | M$ is a TM that accepts string w $\}$

By contradiction, assume $A_{TM}$ is decidable.

then, we have a TM-decider Acceptance for $A_{TM}$.

Acceptance gets on the tape $<M,w>$, accepts if $M$ accepts $w$.

rejests else.

then, let's create a Weird TM with input $<T>$

1. Run Acceptance on input $<T,T>$
2. if Acceptance accepts $<T,T>$, reject
3. if Acceptance rejects $<T,T>$, accept

then, what happens

pandox

$\Rightarrow$ Acceptance decider can't exist

when Weird is run on input $<\text{Weird}>$

accepts if Acceptance rejects $<\text{Weird},\text{Weird}>$

rejects if Acceptance accepts $<\text{Weird},\text{Weird}>$

i.e. if $\text{Weird}$ with input $<\text{Weird}>$ does not accept

return not acceptance $<T,T>$

i.e. if $\text{Weird}$ with input $<\text{Weird}>$ accepts

returns true if $M$ accepts $w$

else returns false
The Halting Problem

**Thm 4.22:** A language $L$ is decidable iff $L$ is Turing-recognizable and $\overline{L}$ is Turing-recognizable (we say that $L$ is co-Turing-recognizable).

$\implies$:
- $L$ is decidable (know how to say YES and NO)
- i.e. every string $w \in L$ is accepted by some TM $M$
- every string $w \notin L$ is rejected
- then $\exists$ TM for $L$ (namely $M$)
- and $\exists$ TM for $\overline{L}$ (namely, create $M'$ equal to $M$ but switch accept & reject states)

$\iff$:
- we have a TM $M_1$ for $L$ and $M_2$ for $\overline{L}$
- create $M_3$, a TM for $L$ that halts on every input (a TM-decider)
  - **sketch:** 1. run $M_1$, if it accepts $w$, then accept
  - 2. else, run $M_2$, if it accepts $w$, then reject
  - **problem:** infinite computation of $M_3$
  - **solution:** a solution
  - 1. run $M_1$, $M_2$ on $w$ in parallel (alternate one step of $M_1$, one step of $M_2$)
  - 2. if $M_1$ accepts, accept
  - 3. if $M_2$ accepts, reject

**Cor 4.23:** $A_{TM}^\#$ is not Turing-recognizable.