Nonregular languages

Which of these languages are regular?

- $B = \{ 0^n1^n \mid n \geq 0 \}$ not regular — we'll see why soon
- $C = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 0's and 1's} \}$ not regular
- $D = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 01's and 10's as substrings} \}$
  
  regular: $0(01)^*0 \cup (01)^*1 \cup \epsilon \cup 01$
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Proof by closure properties: [not in the book]
Suppose we have a DFA with \( p \) states.

Suppose there is a string of length \( \geq p \) that is accepted. Are there other strings that are accepted?

0) \( s = xyz \)

- \( xy^i z \) also accepted

1) \( xy^i z \) is accepted

- \( \forall i \geq 0 \)

2) \( y \neq \epsilon \) or \( |y| > 0 \)

3) \( |xy| \leq p \)
Thm 1.70 [pumping lemma]:

Let $A$ be a regular language. Then there exists a number $p$ s.t.
for every string $s \in A$ of length $\geq p$ there exist strings $x, y,$
and $z$ s.t.

0. $s = xyz,$

1. For each $i \geq 0$, $xy^iz \in A,$

2. $|y| > 0$, and

3. $|xy| \leq p.$
Example: \( B = \{ 0^n1^n \mid n \geq 0 \} \)

Proof that \( B \) is not regular:

By contradiction, suppose that \( B \) is regular. Then, let \( p \) be the PL number for \( B \).

Consider \( s = 0^p1^p \) must be in \( B \) \( \checkmark \) must be of length \( \geq |p| \) \( \checkmark \)

We want to show that there is a problem w. the PL conditions for \( s \):

We need to say that for no \( x, y, z \) the conditions 0)–3) hold simultaneously.

Suppose we have \( x, y, z \) s.t. 0)–3) all hold:

by 3): \( |xy| \leq p \) i.e. \( xy \) contain only 0's

2): \( y \neq \epsilon \) i.e. \( y \) contains at least one 0

1): consider \( i = 2 \): \( xy'y^2z = xyyz \) contains \( p \) 1's

\( \checkmark \) contradiction, hence \( B \) cannot be regular
Example: \( C = \{ w \mid w \text{ has equal number of 0's and 1's} \} \)

Proof that \( C \) is not regular:

By contradiction, assume \( C \) is regular, let \( p \) be the PL number for \( C \).

Consider \( s = \overbrace{0^p1}^p \in C \)
\( \overbrace{0^p0}^p \in C \) is short
\( \overbrace{0^p1^p}^p \in C \)
\( |s| = 2p \geq p \)

Assume we have \( x^i y z \) for which 0)–3) hold:

\( |x|, |y| \leq p \)
\( x, y \) contain only 0's
\( y \) contains at least 1 zero
\( i = 2 \): \( x y z = x y y z \)
\( |x y z| = p + 1 \geq p \\
|s| = 2p \geq p \)

\( C \) is not regular
Pumping lemma for regular lang.

Example:  $F = \{ \text{ww} \mid w \in \{0,1\}^* \}$

Proof that $F$ is not regular:

By contradiction, assume $F$ is regular. Let $p$ be the PL number for $F$.

Consider $s = 0p1p01p$ too short

$0p1p01p \in F \sqrt{1} \sqrt{1} \sqrt{1} \sqrt{1} \sqrt{1} \sqrt{1} \sqrt{1} \sqrt{1} \sqrt{1}$

Suppose $x,y,z$ satisfy (0)-(3):

by (3): $|xy| = p \quad \text{i.e. } x,y \text{ contain only 0's}$
by (2): $y \notin F \quad \text{i.e. } y \text{ contains at least one } 0$
by (1): let $i = 0$ then $xy^iz = xz = 0^k1p01p \text{ where } k < p$
$\text{ & } F \not\ni \text{ F is not regular}$
Pumping lemma for regular lang.

Example: $D = \{ 1^k \mid k \geq 0 \text{ is a square} \}$
Pumping lemma for regular lang.

Example: \( E = \{ 0^i 1^j \mid i > j \} \)