Reductions: if we can reduce (transform) problem A into a problem B, then solving problem B gives solution to problem A.

Example: \(\text{HALT}_{TM} = \{ <M, w> \mid M \text{ is a TM that halts on } w \}\)

Thm 5.1: \(\text{HALT}_{TM}\) is undecidable.

Note: \(\text{HALT}_{TM}\) is the halting problem, \(A_{TM}\) is the acceptance problem.
Thm 5.1: $E_{TM}$ is undecidable, where

$$E_{TM} = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset \}$$

by contradiction, suppose $E_{TM}$ is decidable, i.e. we have a

```
func decide $E_{TM}$ (M)
    \{ returns $TRUE$ if $L(M) = \emptyset$
    \{ $FALSE$ otherwise
```

func decide $HALT_{TM}$ (M, W):

create a TM $M_2$ as follows:
1. erase the input tape
2. write W on the input tape
3. run $M$ on W, if $M$ gets to either the accepting or the rejecting state, accept

if decide $E_{TM}$ ($M_2$): return $FALSE$
else: return $TRUE$

Notice:

$L(M_2) = \{ \emptyset \}$ if M does not
\$ \Sigma^* \$ if M halts on W

if $E_{TM}$ is decidable, then $HALT_{TM}$ is decidable. Therefore, since $\chi$ is undecidable,

$E_{TM}$ is also undecidable.
Thm 5.3: \( \text{REGULAR}_{\text{TM}} \) is undecidable, where

\[
\text{REGULAR}_{\text{TM}} = \{ <M> \mid M \text{ is a TM and } L(M) \text{ is regular} \}
\]
Thm 5.4: $EQ_{TM}$ is undecidable, where

$$EQ_{TM} = \{ <M_1, M_2> \mid M_1, M_2 \text{ are TM's and } L(M_1) = L(M_2) \}$$

Suppose we have \texttt{decideEQ}_{TM}(M_1, M_2) \begin{cases} \text{True} & \text{if } L(M_1) = L(M_2) \\ \text{False} & \text{otherwise} \end{cases}

We want:

\texttt{decideEQ}_{TM}(M)

\begin{align*}
&\text{create } M_2 \text{ s.t. } L(M_2) = \emptyset \\
&\text{if } \texttt{decideEQ}_{TM}(M_1, M_2): \text{ return True if } L(M) = \emptyset \\
&\text{else return False}
\end{align*}
Thm 5.4: \( \text{ALL}_{\text{CFG}} \) is undecidable, where

\[
\text{ALL}_{\text{CFG}} = \{ <G> \mid G \text{ is a CFG and } L(G) = \Sigma^* \}
\]
Rice’s Thm [Problem 5.28]:

Let p be a language property. If p holds for some but not all languages, then the following language is undecidable:

\[ R = \{ \langle M \rangle \mid M \text{ is a TM and } L(M) \text{ satisfies } p \} \]