Nonregular languages

Which of these languages are regular?

- \( B = \{ 0^n1^n \mid n \geq 0 \} \) - not regular, proved last class (directly by contradiction, as well as by Myhill-Nerode)
- \( C = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 0's and 1's} \} \)
- \( D = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 01's and 10's as substrings} \} \)

equivalence classes of \(=_{\text{c}}\):

\[
\begin{align*}
W & \rightarrow |\# \text{of 0's in } W - \# \text{of 1's in } W| = k \\
V & \rightarrow |\# \text{of 0's in } V - \# \text{of 1's in } V| = \ell
\end{align*}
\]

assume \(k \neq \ell\)

then we claim:

\( W \neq e \times V \)

because: take \( z = \begin{cases} 1^k & \text{if } k \geq 0 \\ 0^k & \text{if } k < 0 \end{cases} \)

\( \Rightarrow \) by the Myhill-Nerode Thm: \( C \) is not regular
Which of these languages are regular?

- \( B = \{ 0^n1^n \mid n \geq 0 \} \) - not regular

- \( C = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 0's and 1's} \} \) - we'll show not regular

- \( D = \{ w \in \{0,1\}^* \mid w \text{ has equal number of 01's and 10's as substrings} \} \) - is regular because every occurrence of 01 has to be followed by 10 (or the end of the string)
  \[ \Rightarrow \] # of 01's and # of 10's differ by \( \leq 1 \)
  \[ \Rightarrow \] have a FA

Proof by closure properties:

we know that \( B \) is not regular

by contradiction, suppose that \( C \) is regular

\[ B = C \cap 0^*1^* \]

Regular languages are closed under \( \cap \)
(by the Cartesian product construction)

\[ \text{Thus, this is regular} \Rightarrow \text{but we know that } B \text{ is not regular} \Rightarrow \]
Pumping lemma for regular lang.

Suppose we have a DFA with $p$ states.

Suppose there is a string of length $\geq p$ that is accepted. Are there other strings that are accepted?

Example:

- $i=1$: $xy^i z = xyz$ accepted
- $i=2$: $xy^i z = xyxz$ accepted
- $i=3$: $xy^i z = xyyz$ accepted

Also notice that $|xy| \leq p$ because after $p$ symbols we go through $p+1$ states $\Rightarrow$ have to have a repetition of a state = loop

$S = xyz$

$x = bba$

$y = abaa$

$z = ba$

Notice that $xy^i z$ are also accepted $\forall i \geq 0$

$y \notin \varepsilon$ since the transitions on the loop (yellow) are labeled by symbols (not $\varepsilon$)
Pumping lemma for regular lang.

Thm 1.70 [pumping lemma]:

Let $A$ be a regular language. Then there exists a number $p$ s.t. for every string $s \in A$ of length $\geq p$ there exist strings $x, y,$ and $z$ s.t.

0. $s = xyz$,
1. For each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

we will show that a language $L$ is not regular by finding a contradiction w. the statement of the PL:

**outline:**

by contradiction, assume $L$ is regular, thus the PL holds, let $p$ be the PL number for $L$

we'll find a string $s \in L$ (s.t. $|s| \geq p$ and $s$ cannot be decomposed into $xyz$ satisfying 0-3)
Example: \( B = \{ 0^n1^n \mid n \geq 0 \} \)

by contradiction, assume \( B \) is regular, let \( p \) be the number from the PL applied on \( B \)

consider \( s = 0^p1^p \in B \) \( \checkmark \) \( |s| = 2p > p \) \( \checkmark \)

Suppose we have \( x, y, z \) satisfying 0)-3)

\[ s = 0 \cdots 0 1 \cdots 1 \]

by 0) \( s = xyz \)

2) \( y \neq \epsilon \Rightarrow y \) contains at least one zero

3) \(|xy| \leq p \Rightarrow xy \) contain only zeros

1) \( \forall i \geq 0 : \ xy^i z \in B \)

\[ \text{consider } i = 0 : \ xy^i z = xz = 0^p \cdot 1^p \in B \]

by contradiction, the PL does not hold for \( B \), thus \( B \) is not regular.
Example: \( C = \{ w \mid w \text{ has equal number of 0's and 1's} \} \)

Suppose, by contradiction, that \( C \) is regular. Then PL holds for \( C \), let \( p \) be the PL number for \( C \).

Consider \( s = 0^p1^p \) \( s \in C \) \( |s| = 2p \geq p \)

Suppose there exist \( x, y, z \) satisfying (0)-(3)

Continue with the argument (\( \star \)) from the last slide (replace \( B \) with \( C \))
Example: \( F = \{ \text{ww} \mid w \in \{0,1\}^* \} \)

...
Pumping lemma for regular lang.

Example: \( D = \{ 1^k \mid k \geq 0 \text{ is a square} \} \)

We will show that \( D \) is not regular.

By contradiction, assume \( D \) is regular and let \( p \) be the PL number for \( D \).

Consider \( s = 1^{p^2} \in D \), \( |s| = p^2 \geq p \).

Imagine \( x, y, z \) satisfying 0)–3) would exist.

Then, by 0) \( s = xy^2 \)

2) \( |y| > 0 \)

3) \( |xy| \leq p \)

1) \( \forall i \geq 0 : xy^iz \in D \)

Let's see what happens with \( i = 2 \): \( xy^2z = xyy^2 = 1^{p^2 + |y|} \)

But we know:

\[ 0 < |y| \leq p \]

\[ (p+1)^2 = p^2 + 2p + 1 \]

Thus \( p^2 < p^2 + |y| < (p+1)^2 \)

Thus, \( 1^{p^2 + |y|} \notin D \).
Pumping lemma for regular lang.

Example: \( E = \{ 0^i1^j \mid i > j \} \)

By contradiction, assume \( E \) regular, let \( p \) be the PL number for \( E \).

Consider \( S = \{ 0^i1^j \mid \text{not in } E \} \):
- \( 0^p1^p \) - possibly shorter than \( p \)
- \( 0^p \) - \( \notin E \), \( \text{1st bit} \) - but does not yield a contradiction in the PL
- \( 0^p1^p \) - \( \notin E \), \( \text{1st bit} \)

Imagine there are \( x,y,z \) satisfying 0)-3)

\( S = \begin{array}{cccc}
0 & \cdots & 0 & 1 \\
\vdots & & \vdots & \vdots \\
& 1 & \cdots & 1 \\
\end{array} \)

by 0) \( s = xy^2z \)
2) \( |y| > 0 \)
3) \( |xy| \leq p \) - all zeros
1) \( \forall i \geq 0: \ xy^i z \in E \)

Consider \( i = 2 \): \( xy^2z \) → increases \( \#0's \) \( \notin E \)
\( i = 0 \): \( xy^0z = xz \) → decreases \( \#0's \) by at least 1
\( \notin E \)