Determinism: computation always continues in a uniquely determined way.

Nondeterminism: have more (or none) choices

Example:

\[ L = \{ w \in \{0,1\}^* \mid w \text{ contains 001 or 0101 as a substring} \} \]

\[ A = \{ w \in \{0,1\}^* \mid w \text{ contains 001 as a substring} \} \]

\[ B = \{ w \in \{0,1\}^* \mid w \text{ contains 0101 as a substring} \} \]

\[ L = A \cup B \]

Acceptance: if there exists a computation path leading (and ending in) an accepting state.

1001100110 is accepted

011110110 is not accepted
Nondeterminism

Determinism: computation always continues in a uniquely determined way.

Nondeterminism: have more (or none) choices

Example:

\[ \{ w \in \{0,1\}^* \mid w \text{ contains } 001 \text{ or } 0101 \text{ as a substring} \} \]

Nondeterministic FA can also use \( \varepsilon \)-transitions:
Nondeterminism

Example:

\[ \{ w \in \{0,1\}^* \mid w \text{ contains 1 in the third position from the end} \} \]

Does there exist a (deterministic) FA recognizing this language?

There is a DFA with 8 states (the 8 on the right) but not fewer.
Nondeterminism

Example:

\( \{ w \in \{0\}^* \mid |w| \text{ is divisible by 2 or 3} \} \)

- \( Q = \{ q_0, q_1, \ldots, q_5 \} \)
- \( q_0 = q_0 \)
- \( F = \{ q_2, q_4 \} \)
- \( \Sigma = \{ \varepsilon \} \)
- \( \delta(q_0, \varepsilon) = \{ q_2 \} \)
- \( \delta(q_0, 0) = \{ q_4 \} \)
- \( \delta(q_2, 0) = q_4 \)
Nondeterminism

Formal definition:

A **nondeterministic finite automaton** (NFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- $Q$ is a finite set of states
- $\Sigma$ is a (finite) alphabet
- $\delta: Q \times \Sigma \cup \{\varepsilon\} \rightarrow \mathcal{P}(Q)$ is the transition function
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept state
Let $N=(Q, \Sigma, \delta, q_0, F)$ be an NFA and let $w=w_1w_2...w_n$ where each $w_i \in \Sigma$. Then $N$ accepts $w$ iff there exists a sequence of states $r_0, r_1, r_2, ..., r_n$ such that:

- $r_0 = q_0$
- $\{r_{k+1}\} \subseteq \delta(r_k, \sigma_{k+1})$ or, equivalently: $r_{k+1} \in \delta(r_k, \sigma_{k+1})$ \(\forall k \in \{0, 1, 2, ..., n-1\}\)
- $r_n \in F$

**OK for NFA's with no $\varepsilon$-transitions**

If we have $\varepsilon$-transitions, change the top of the slide to:

$N$ accepts $w$ iff there exist $\sigma_1, ..., \sigma_n \in \Sigma_\varepsilon$ s.t. $w=\sigma_1\sigma_2...\sigma_n$ and there exists a sequence of states $r_0, r_1, ..., r_n$ \(\forall i\in\{0, 1, ..., n\}: r_i \in Q\) such that:

- $r_0 = q_0$
- $r_n \in F$
Thm 1.39: Every NFA has an equivalent DFA.

Proof idea:
- for starters, no ε-transitions in the NFA
- example:

\[ \text{have an NFA} \]
\[ N = (Q, \Sigma, \delta, q_0, F) \]
\[ \text{want: a DFA } M = (Q_M, \Sigma, \delta_M, q_0, F_M) \text{ s.t. } L(N) = L(M) \]
\[ \text{let } Q_M = \mathcal{P}(Q) \]
\[ q_M = \{q_0\} \]
\[ F_M = \{ S \subseteq Q_M \mid S \cap F \neq \emptyset \} \]
\[ \delta_M(S, c) = \bigcup_{q \in S} \delta(q, c) \]
\[ \forall S \subseteq Q_M, \forall c \in \Sigma \]
Thm 1.39: Every NFA has an equivalent DFA.

Proof idea, part 2 (getting rid of \(\varepsilon\)-transitions in the NFA):
- for \(R \subseteq Q\) let:

\[
E(R) = \{ q \in Q \mid q \text{ can be reached from } R \text{ by traveling along 0 or more } \varepsilon\text{-arrows} \}
\]
Thm 1.39: Every NFA has an equivalent DFA.

Proof idea, part 2 (getting rid of ε-transitions in the NFA):
- for $R \subseteq Q$ let:

$$E(R) = \{ q \in Q \mid q \text{ can be reached from } R \text{ by traveling along } 0 \text{ or more } \varepsilon\text{-arrows} \}$$

Example:

Have an NFA

$$N = (Q, \Sigma, \delta, q_0, F)$$

want: a DFA $M = (Q_M, \Sigma, \delta_M, q_M, F_M)$ s.t. $L(N) = L(M)$

let

$$Q_M = \mathcal{P}(Q)$$

$$q_M = \{ q_0 \} \cup E(\{ q_0 \})$$

$$F_M = \{ S \in Q_M \mid S \cap F \neq \emptyset \}$$

$$\delta_M(s, \sigma) = E(\bigcup_{q \in S} \delta(q, \sigma)) \quad \forall s \in Q_M \forall \sigma \in \Sigma$$
Thm 1.45 (revisited): The class of regular languages is closed under the union operation.

Suppose we have two regular languages $L_1, L_2$; want to show that $L_1 \cup L_2$ is regular.

Let $N_1, N_2$ be NFAs for $L_1, L_2$; we will construct NFA $N$ s.t. $L(N) = L(N_1) \cup L(N_2)$.

**Diagram:**

1. **$N_1$:**
   - $Q = Q_1 \cup Q_2 \cup \{q_0\}$
   - $q_0 = q_0^1$
   - $F = F_1 \cup F_2$
   - $\delta(q, a) = \begin{cases} \delta_1(q, a) & \text{if } q \in Q_1, a \in \Sigma_1, \varepsilon \in \Sigma_1, a \in \Sigma_1, a \in \Sigma_1, \varepsilon \in \Sigma_1, a \in \Sigma_1, a \in \Sigma_1, \varepsilon \in \Sigma_1, a \in \Sigma_1, a \in \Sigma_1, \varepsilon \end{cases}$

2. **$N_2$:**
   - $Q = Q_1 \cup Q_2 \cup \{q_0\}$
   - $q_0 = q_0^2$
   - $F = F_1 \cup F_2$
   - $\delta(q, a) = \begin{cases} \delta_2(q, a) & \text{if } q \in Q_2, a \in \Sigma_2, \varepsilon \in \Sigma_2, a \in \Sigma_2, a \in \Sigma_2, \varepsilon \in \Sigma_2, a \in \Sigma_2, a \in \Sigma_2, \varepsilon \in \Sigma_2, a \in \Sigma_2, a \in \Sigma_2, \varepsilon \end{cases}$

Let $N = (Q, \Sigma, \delta, q_0, F)$ such that $L(N) = L(N_1) \cup L(N_2)$. 

**Diagram:**

- $N_1$: Start state $q_0$, accepting states $F_1$.
- $N_2$: Start state $q_0$, accepting states $F_2$. 

**Conclusion:**

The class of regular languages is closed under the union operation.
Thm 1.47: The class of regular languages is closed under the concatenation operation.

have two NFAs $N_1, N_2$

want: an NFA $N$ s.t. $L(N) = L(N_1) \cdot L(N_2)$
**Thm 1.49:** The class of regular languages is closed under the star operation.

Suppose $L$ is a regular language, we want to show that $L^*$ is regular.

Let $N$ be a NFA for $L$, we will construct $N'$ for $L^*$.

Let $N = (Q, \Sigma, \delta, q_0, F)$

Assume $q_i \notin Q$

Let $N' = (Q', \Sigma, \delta', q'_0, F')$

where

$Q' = Q \cup \{ q'_i \}$

$q'_i = q_i$

$F' = \{ q'_i \} \cup F$

$$
\delta'(q'_i, c) = \begin{cases} 
\delta(q_i, c) & \text{if } q_i = q'_i, c \in \Sigma \\
\delta(q_i, \epsilon) & \text{if } q_i = q'_i, c = \epsilon \\
\delta(q_i, c) & \text{if } q \in Q, c \in \Sigma \\
\delta(q_i, c) & \text{if } q \in F, c \in \Sigma \\
\{q_i\} & \text{if } c = \epsilon
\end{cases}
$$

$\{q'_0\} \cup \delta(q'_i, \epsilon)$