Sequential Dynamical Systems

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University of Rochester
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\[ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \]

\[ G = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ F = \begin{pmatrix} \emptyset & \emptyset & \emptyset & \emptyset \end{pmatrix} \]
Dynamical Systems

- Transportation systems
- Internet
- National power grid & markets
- Mobile (ad-hoc) systems
- Supply chain management
- Etc.
Typical Questions

- **Predict behavior**
  - Avg. commute time during office hours
  - Average round-trip time for IP packets

- **Control behavior**
  - Reduce vehicular congestion
  - Reduce internet traffic congestion

- **Make policy decisions**
  - Add an extra lane on highway?
  - Increase bandwidth?
Simulation

- Mimics behavior of some other (real-world) system
- Dynamics generator
- Models **global** dynamic behavior using **local** mappings
Sequential Dynamical Systems

\[ \pi = \begin{array}{cccccc}
0 & 0 & 1 & 0 & 0 & 1
\end{array} \]
Sequential Dynamical Systems

SDS $S = (G, F, \pi)$
Related Models

- Cellular Automata
- Graph Automata
- Communicating (Hierarchical) Finite State Machines
- Discrete Hopfield Networks
Outline

- Phase Space of SDS
- Basic Computational Problems
- Predecessor Existence
- Permutation Existence
- Reachability
Phase Space of SDS

\[ \pi = \begin{bmatrix} 0011 & 1101 & 1111 \\ 0101 & 0111 & 1011 \\ 1110 & 0001 & 1000 \\ 0110 \\ 1100 \\ 0010 \\ 0100 \end{bmatrix} \]
Basic Problems

- Predecessor Existence
- Garden-of-Eden Points
  - Number of GoE Points
- Permutation Existence
- Reachability
- Fixed Point
  - Number of Fixed Points
Predecessor Existence

Given SDS $S = (G, F, \pi)$, and a configuration $C'$, is there a configuration $C$ such that $C \rightarrow C'$ in one step in $S$?
Predecessor Existence

**EASY Cases**

<table>
<thead>
<tr>
<th>Domain</th>
<th>Function</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Field</td>
<td>Lin. Comb.</td>
<td>ANY</td>
</tr>
<tr>
<td>Boolean</td>
<td>{XOR, XNOR}</td>
<td>ANY</td>
</tr>
<tr>
<td>Boolean</td>
<td>Symmetric</td>
<td>Bounded Treewidth</td>
</tr>
</tbody>
</table>

Spot Quiz: Spot a corollary!!
Predecessor Existence

HARD (NP-complete) Cases

<table>
<thead>
<tr>
<th>Domain</th>
<th>Function</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boolean</td>
<td>Simple threshold</td>
<td></td>
</tr>
<tr>
<td>Boolean</td>
<td>Exact-k</td>
<td></td>
</tr>
<tr>
<td>Boolean</td>
<td>{OR, AND}</td>
<td></td>
</tr>
<tr>
<td>Boolean</td>
<td></td>
<td>Planar</td>
</tr>
</tbody>
</table>
NP-hardness of PRE for \{\text{OR, AND}\} - SDS

MONOTONE-3SAT = \{ \phi \mid \phi \in 3SAT \text{ and each clause is either of the form } (x_1 \land x_2 \land x_3) \text{ or of the form } (: x_1 \lor : x_2 \lor : x_3) \}

\phi = (x_1 \land x_2 \land x_3) \land ( : x_2 \lor : x_3 \lor : x_4)

\sigma(\phi) = a \rightarrow x_1 \rightarrow c_1 \rightarrow b \lor x_4 \rightarrow x_2 \rightarrow c_2 \rightarrow d \land b \rightarrow OR \land c_2 \rightarrow \text{AND}
\[ \phi = (x_1 \land x_2 \land x_3) \forall \exists (x_2 \land x_3 \land x_4) \]
\[ \sigma(\phi) = \]

<table>
<thead>
<tr>
<th>( \pi )</th>
<th>( C' )</th>
<th>( C \Rightarrow C' ) iff ( C ) is:</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>d</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( c_2 )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( x_i )</td>
<td>1</td>
<td>Truth value of ( x_i ) in a satisfying ass. of ( \phi )</td>
</tr>
</tbody>
</table>

Quiz: Am I being sloppy?
Permutation Existence

Given SDS $G$, $F$ and a configurations $C$ and $C'$, is there a permutation $\pi$ such that $C \rightarrow C'$ in one step in $S=(G,F,\pi)$?
## Permutation Existence

### EASY Cases

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<thead>
<tr>
<th>Domain</th>
<th>Function</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boolean</td>
<td>NOR</td>
<td>ANY</td>
</tr>
<tr>
<td>Boolean</td>
<td>NAND</td>
<td>ANY</td>
</tr>
<tr>
<td>Boolean</td>
<td>OR</td>
<td>ANY</td>
</tr>
<tr>
<td>Boolean</td>
<td>AND</td>
<td>ANY</td>
</tr>
</tbody>
</table>
Generalized Permutation Existence

Given SDS $G$, $F$, and configurations $C$ and $C'$ ($C'$ may contain don’t-care nodes), is there a permutation $\pi$ such that in $S = (G, F, \pi)$, $C'$ and $\text{next}(C)$ coincide on all non-don’t-care nodes?

\[
\begin{array}{ccc}
G &=& \begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{array} \\
C_1 &=& \begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{array} \\
C_2 &=& \begin{array}{cccc}
1 & X & X & X \\
1 & X & X & X \\
1 & X & X & X \\
1 & X & X & X \\
\end{array}
\end{array}
\]
### Generalized Permutation Existence

#### HARD (NP-complete) Cases

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<thead>
<tr>
<th>Domain</th>
<th>Function</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boolean</td>
<td>Simple Thresh.</td>
<td></td>
</tr>
<tr>
<td>Boolean</td>
<td>NAND</td>
<td>Bounded degree</td>
</tr>
<tr>
<td>Boolean</td>
<td>NOR</td>
<td>Bounded degree</td>
</tr>
<tr>
<td>Boolean</td>
<td>Symmetric</td>
<td>Planar</td>
</tr>
</tbody>
</table>
**Generalized Permutation Existence**

**OPEN Cases**

<table>
<thead>
<tr>
<th>Domain</th>
<th>Function</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boolean</td>
<td>AND</td>
<td></td>
</tr>
<tr>
<td>Boolean</td>
<td>OR</td>
<td></td>
</tr>
</tbody>
</table>
Reachability

Given SDS $S = (G, F, \pi)$, and configurations $C$ and $C'$, is there a path from $C$ to $C'$ in the phase space of $S$?
## Reachability

### EASY Cases

<table>
<thead>
<tr>
<th>Domain</th>
<th>Function</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boolean</td>
<td>Edge-Symmetric Weighted Threshold</td>
<td>ANY</td>
</tr>
<tr>
<td>Boolean</td>
<td>Symmetric and Monotone</td>
<td>ANY</td>
</tr>
</tbody>
</table>
Reachability

**HARD (PSPACE-complete) Cases**

<table>
<thead>
<tr>
<th>Domain</th>
<th>Function</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boolean</td>
<td>Simple Inverted Threshold</td>
<td></td>
</tr>
<tr>
<td>Boolean</td>
<td>Symmetric</td>
<td>d-regular</td>
</tr>
<tr>
<td>Boolean</td>
<td>Asymmetric Weighted Threshold</td>
<td></td>
</tr>
</tbody>
</table>
Edge-Symmetric Weighted Threshold

\[ f_1 = 1 \text{ iff } (w_1^1 s_1 + w_2^1 s_2 + w_3^1 s_3 > k_1) \]

\[ f_2 = 1 \text{ iff } (w_1^2 s_1 + w_2^2 s_2 + w_3^2 s_3 > k_1) \]

Edge-symmetric: \( w_2^1 = w_1^2 \),
so think of edge being associated with edges
Poly-Time Algorithm for REACH

- Any configuration reaches a fixed point in polynomial number of steps, when the functions are edge-symm. weighted threshold
- Thus, REACH 2 P
- Proof is via potential argument
  - Define (nonnegative) potential for each configuration
  - Prove: initial configuration has “small” potential
  - Prove: each step decreases potential by a “significant” amount
Potential for Configuration C

\[ P(v,C) = \begin{cases} 
  k_v & \text{if } C(v) = 1 \\
  w_v + w_1 + w_2 - k_v & \text{if } C(v) = 0
\end{cases} \]

\[ G = w_2 \]

\[ P(e_1,C) = \begin{cases} 
  w_1 & \text{if } C(v) \neq C(v_1) \\
  0 & \text{if } C(v) = C(v_1)
\end{cases} \]
Potential Change While Updating a Node

$P(v, C) = k_v$ if $C(v) = 1$

$= w_v + w_1 + w_2 - k_v$ if $C(v) = 0$

$P(e_1, C) = w_1$ if $C(v) \neq C(v_1)$

$= 0$ if $C(v) = C(v_1)$

<table>
<thead>
<tr>
<th></th>
<th>$v$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$P(v)$</th>
<th>$P(e_1)$</th>
<th>$P(e_2)$</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Before</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$w_v + w_1 + w_2 - k_v$</td>
<td>$w_1$</td>
<td>0</td>
<td>$w_1 &gt; k_v$</td>
</tr>
<tr>
<td>After</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$k_v$</td>
<td>0</td>
<td>$w_2$</td>
<td>Potential +</td>
</tr>
</tbody>
</table>
Conclusion

- Dynamical systems: large and unpredictable
- SDS: formal model of simulations
- Easy and hard cases for
  - Predecessor Existence
  - Permutation Existence
  - Reachability
Acknowledgments

- Research presented here done mostly by Madhav Marathe and his group at LANL
- Thanks to Madhav Marathe for helpful advice