Chapter 2

Folkman Number $F_e(3, 3; 4)$

2.1 Introduction

Given a simple graph $G$, we write $G \rightarrow (a_1, \ldots, a_k)^e$ and say that $G$ arrows $(a_1, \ldots, a_k)^e$ if for every edge $k$-coloring of $G$, a monochromatic $K_{a_i}$ is forced for some color $i \in \{1, \ldots, k\}$. Likewise, for graphs $F$ and $H$, $G \rightarrow (F, H)^e$ if for every edge 2-coloring of $G$, a monochromatic $F$ is forced in the first color or a monochromatic $H$ is forced in the second. Define $F_e(a_1, \ldots, a_k; p)$ to be the set of all graphs that arrow $(a_1, \ldots, a_k)^e$ and do not contain $K_p$; they are often called Folkman graphs. The edge Folkman number $F_e(a_1, \ldots, a_k; p)$ is the smallest order of a graph that is a member of $F_e(a_1, \ldots, a_k; p)$.

In 1967, Erdős and Hajnal [21] asked the question: Does there exist a $K_4$-free graph that is not the union of two triangle-free graphs? This question is equivalent to asking for the existence of a $K_4$-free graph such that in any edge 2-coloring, a monochromatic triangle is forced. After Folkman proved the existence of such a graph, the question then became to find how small this graph could be, or using the above notation, what is the value of $F_e(3, 3; 4)$. Prior to this work, the best known bounds for this case were $19 \leq F_e(3, 3; 4) \leq 941$ [70, 19].

In general, an upper bound of $F_e(s, t; k)$ is considered easier to determine than a lower bound. This is due to the upper bound requiring one $K_k$-free witness that arrows $(s, t)$, while an improvement to the lower bound requires a proof that all graphs of a given order do not arrow $(s, t)$. This is perhaps a reason for the puzzling large range between the lower and upper bounds of $F_e(3, 3; 4)$.

Table 2.1 summarizes known results for $F_e(3, 3; k)$. Since the Ramsey number $R(3, 3) = 6$, it follows that $F_e(3, 3; k) = 6$ for $k \geq 7$. In 1968, Graham [32] responded to Erdős and Hajnal by presenting an explicit $K_6$-free graph on 8 vertices that arrows $(3, 3)$. As no such graph exists with 7 vertices, this showed $F_e(3, 3; 6) = 8$. This graph, $K_8 - C_5$, is displayed in Figure 2.1, and a summary of the proof that $K_8 - C_5 = K_3 + C_5 \rightarrow (3, 3)$ is found in Theorem 1.

**Theorem 1** (Graham [32]). $G = K_8 - C_5 = K_3 + C_5 \rightarrow (3, 3)$
Proof. Assume there is an edge coloring of $G$ such that neither of the colors contain a triangle; call the parts of this coloring $R$ (red) and $B$ (blue). Consider the triangle of $G$ that is joined to $C_5$. Two of the vertices in this triangle will be incident to a red and blue edge, as the $K_3$ is non-monochromatic. Let one of those vertices be $v$. This vertex will be adjacent to all five vertices $c_1, \ldots, c_5$ of the $C_5$. At least 3 of these edges will be one color, so with out loss of generality, say $\{v, c_1\}, \{v, c_2\}, \{v, c_3\} \in B$. Two of $\{c_1, c_2, c_3\}$ must be adjacent, say $\{c_1, c_2\} \in E(C_5)$. Clearly, $\{c_1, c_2\}$ must be red to avoid a blue triangle. Pick $u \in V(K_3)$ such that $\{u, v\} \in B$. Then, neither $\{u, c_1\}$ nor $\{u, c_2\}$ can be in $B$ or a blue triangle is formed. However, if they are both red, then they form a red triangle with $\{c_1, c_2\}$. Therefore, a monochromatic triangle must exist. 

Table 2.1: Known values and bounds for $F_e(3, 3; k)$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$F_e(3, 3; k)$</th>
<th>Graphs</th>
<th>Who</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\geq 7$</td>
<td>6</td>
<td>$K_6$</td>
<td>folklore</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>8</td>
<td>$K_3 + C_5$</td>
<td>Graham 1968</td>
<td>[32]</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>659 graphs</td>
<td>Piwakowski et al. 1999</td>
<td>[66]</td>
</tr>
<tr>
<td>4</td>
<td>$19 - 786$</td>
<td>$L_{786}$</td>
<td>RX 2007, this work 2012</td>
<td>[70]</td>
</tr>
</tbody>
</table>

The case for $k = 5$ received much attention up until 1999, when Piwakowski et al. determined that $F_e(3, 3; 5) = 15$ [66]. The first upper bound $F_e(3, 3; 5) \leq 42$ was obtained by Schäuble in 1969 [77], although the proof of existence is credited to an unpublished work by Pósa. In 1971, Graham and Spencer [33] improved the bound to $F_e(3, 3; 5) \leq 23$. Both constructions rely on cleverly connecting a number of $C_5$ graphs and a triangle. The bound was then improved to 18 by Irving in 1973 [41], 16 by Hadziivanov and Nenov in 1979 [35], and 15 by Nenov in 1981 [62]. The latter two results were published in Russian and seemed to go unnoticed for some time.

The computational approach by Piwakowski et al. to determine $F_e(3, 3; 5) \geq 15$ involved processing a large number of graphs to show that no 14-vertex graph exists in $F_e(3, 3; 5)$. 

Figure 2.1: $K_8 - C_5$, the witness of $F_e(3, 3; 6) = 8$
Since $R(3, 5) = 14$, any 14-vertex graph $G \in \mathcal{F}_e(3, 3; 5)$ will contain a $K_3$. They determined a number of properties of $G \setminus K_3$, and all graphs on 11 vertices with these properties were processed in order to reconstruct graphs $G$. However, no such graphs were found.

Figure 2.2 shows the unique bicritical 15-vertex graph in $\mathcal{F}_e(3, 3; 5)$. It is bicritical because (a) adding any edge forms a $K_5$ and (b) removing any edge makes it not arrow $(3, 3)$. This graph plays an important role in the vertex Folkman number $F_v(3, 3; 4)$, as removing vertex $v$ yields the unique bicritical witness of $F_v(3, 3; 4) = 14$.

![Figure 2.2: Only bicritical graph of all 659 witnesses to $F_e(3, 3; 5) = 15$, where $v$ is connected to all other 14 vertices.](image)

The focus of this chapter is on the most studied open Folkman number, $F_e(3, 3; 4)$, and ways the well-known graph MAX-CUT problem can determine arrowing of triangles. The next section overviews the rich history of this number.

## 2.2 History of $F_e(3, 3; 4)$

Table 2.2 summarizes the events surrounding $F_e(3, 3; 4)$, starting with Erdős and Hajnal’s [21] original question of existence. After Folkman [24] proved the existence, Erdős, in 1975, offered $\$100 for deciding if $F_e(3, 3; 4) < 10^{10}$.

Most work on the upper bound of $F_e(3, 3; 4)$ has made use of counting triangles of an edge-colored graph, an idea originally presented by Goodman in 1959 [31], which we briefly describe. Note that there is essentially a single coloring of a non-monochromatic triangle: two edges are one color and one edge is the other. A non-monochromatic triangle therefore has two vertices that are incident to both a red and blue edge. Let $G$ be an edge-colored graph with no monochromatic triangles. Let $t_{RB}(x)$ count the triangles $\{x, y, z\}$ where $\{x, y\}$ is red and $\{x, z\}$ is blue; then, $\sum_{v \in V(G)} t_{RB}(v) = 2t_\Delta(G)$. If $G_x$ is the induced subgraph of $N_G(x)$, then each edge in $G_x$ counts a triangle, yielding $\sum_{v \in V(G)} |E(G_v)| = 3t_\Delta(G)$. Since no monochromatic triangle exists, the vertices of each $G_x$ can be partitioned such that $MC(G_x) = t_{RB}(x)$. Combining these equations gives $\sum_{v \in V(G)} MC(G_v) = \frac{2}{3}|E(G_V)|$.

However, if every coloring of a graph $G$ contains a monochromatic triangle, then some
$G_v$ can not be partitioned completely, resulting in Theorem 2.

**Theorem 2** (Spencer [79]). Let $G$ be a graph and $G_v$ be the graph induced by $N_G(v)$. If

$$\sum_{v \in V(G)} MC(G_v) < \frac{2}{3}|E(G_V)|,$$

then $G \rightarrow (3,3)$.

Deciding $F_e(3,3;4) < 10^{10}$ remained open for over 10 years. Frankl and Rödl [25] nearly met Erdős’ request in 1986 when they showed that $F_e(3,3;4) < 7.02 \times 10^{11}$ using probabilistic arguments and ideas similar to those described above. In 1988, Spencer [79], in a seminal paper also using probabilistic techniques, proved the existence of a Folkman graph of order $3 \times 10^9$ (after an erratum by Hovey) with out explicitly constructing it. The main idea behind his result involved $G = G(n, p)$, the random graph with $n$ vertices and edge probability $p$. From this graph, a $K_4$-free graph $G^*$ is obtained by randomly removing an edge from each $K_4$ in $G$. By setting $n = 3 \times 10^9$, he showed that a $G^*$ satisfying the condition in Theorem 2 exists with positive probability.

Erdős then offered $100 for deciding if $F_e(3,3;4) < 10^6$ (see [9], page 46). Much time passed until 2007, when Lu and Dudek-Rödl independently showed it to be true. Lu determined $F_e(3,3;4) \leq 9697$ by constructing a family of $K_4$-free circulant graphs (which we discuss in Section 2.5) and showing that some such graphs arrow $(3,3)$ using a combination of spectral analysis and Theorem 2. The main idea behind his proof involves a graph $H$ being $\delta$-fair if $MC(H) < (\frac{1}{2} + \delta)|E(H)|$. From Theorem 2 it follows that if each $H_v$ is $\frac{1}{6}$-fair, then $H \rightarrow (3,3)$. Lu was able to show that $d$-regular graphs were $\delta$-fair if the smallest

<table>
<thead>
<tr>
<th>Year</th>
<th>Lower/Upper Bounds</th>
<th>Who/What</th>
<th>Ref.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1970</td>
<td>exist</td>
<td>Folkman</td>
<td>[24]</td>
</tr>
<tr>
<td>1972</td>
<td>10 –</td>
<td>Lin</td>
<td></td>
</tr>
<tr>
<td>1975</td>
<td>$- 10^{10}$?</td>
<td>Erdős offers $100 for proof</td>
<td></td>
</tr>
<tr>
<td>1986</td>
<td>$- 8 \times 10^{11}$</td>
<td>Frankl-Rödl</td>
<td>[25]</td>
</tr>
<tr>
<td>1988</td>
<td>$- 3 \times 10^9$</td>
<td>Spencer</td>
<td>[79]</td>
</tr>
<tr>
<td>1999</td>
<td>16 –</td>
<td>Piwakowski et al. (implicit)</td>
<td>[66]</td>
</tr>
<tr>
<td>2007</td>
<td>19 –</td>
<td>Radziszowski-Xu</td>
<td>[70]</td>
</tr>
<tr>
<td>2008</td>
<td>$- 9697$</td>
<td>Lu</td>
<td>[56]</td>
</tr>
<tr>
<td>2008</td>
<td>$- 941$</td>
<td>Dudek-Rödl</td>
<td>[19]</td>
</tr>
<tr>
<td>2012</td>
<td>$- 786$</td>
<td>this work</td>
<td></td>
</tr>
<tr>
<td>2012</td>
<td>$- 100$?</td>
<td>Graham offers $100 for proof</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.2: Timeline of progress on $F_e(3,3;4)$. 

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$G_v$ appears on page 7 of the document. The text from the page is reproduced here:

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eigenvalue of the adjacency matrix was greater than $-2\delta d$, and found a number of “small”
graphs, including one with order 9697, that had this property.

Dudek and Rödl reduced the upper bound to the best known to date, 941. Their method,
which we have pursued further with some success, is discussed in the next section. During the
2012 SIAM Conference on Discrete Mathematics in Halifax, Nova Scotia, Ronald Graham
announced a $100 award for determining if $F_e(3, 3; 4) < 100$. We discuss a possible witness
for this in Section 2.4.1.

The lower bound for $F_e(3, 3; 4)$ was much less studied than the upper bound. Lin [55]
obtained a lower bound on 10 in 1972. Without the help of a computer, he showed that
$F_e(a_1, \ldots, a_k; R(a_1, \ldots, a_k) - 1) \geq R(a_1, \ldots, a_k) + 4$, giving $F_e(3, 3; 5) \geq 10$. The next
improvement did not come until 1999 when $F_e(3, 3; 5) = 15$ [66] was determined. The
659 graphs on 15 vertices witnessing $F_e(3, 3; 5) = 15$ contain $K_4$, thus giving the bound
16 $\leq F_e(3, 3; 4)$.

In 2007, Radziszowski and Xu gave a computer-free proof of $18 \leq F_e(3, 3; 4)$ and improved
the lower bound further to 19 with the help of computations [70]. A summary of this work
follows.

**Theorem 3** (Radziszowski and Xu [70]). $F_e(3, 3; 4) \geq 18$

**Proof.** To show that $F_e(3, 3; 4) \geq 18$, we must show that no $K_4$-free graph with 17 vertices
arrows $(3, 3)$. Define graph $G_{17}$ as $V(G) = \mathbb{Z}_{17}$ and $E(G) = \{\{u, v\} \mid u - v = \alpha^2\}$, where
$\alpha^2 \in \{1, 2, 4, 8\}$. This circulant graph is the well-known Paley graph of order 17, has no
$K_4$, and is the unique lower-bound witness to $R(4, 4) = 18$. The subgraphs of $G_{17}$ induced
by distances $\{1, 4\}$ and $\{2, 8\}$ do not contain triangles, and therefore $G_{17} \not\in F_e(3, 3; 4)$.
Assume there exists a graph $G \in F_e(3, 3; 4)$; since $G$ is non-isomorphic to $G_{17}$ and does
not contain a $K_4$, it must contain a $K_4$. Connecting the vertices $\{v_1, v_2, v_3, v_4\}$ of this $K_4$
with the other 13 vertices of $G_{17}$ cannot cause a $K_5$, and the resulting graph $G'$ is therefore
in $F_e(3, 3; 5)$. However, since the edges incident to the $K_4$ do not form a triangle with
each other, $G' \setminus \{v_1, v_2, v_3\}$ is also in $F_e(3, 3; 5)$. This contradicts $F_e(3, 3; 5) = 15$ and thus
$F_e(3, 3; 4) \geq 18$.

The proof of $F_e(3, 3; 4) \geq 19$ follows the same idea, but is slightly more complicated due to
a larger number of graphs involved. If a 18-vertex graph $G \in F_e(3, 3; 4)$ exists, then because
$R(4, 4) = 18$, it must contain a $K_4$. Radziszowski and Xu showed that $G \setminus K_4$ must be
isomorphic to one of the 153 14-vertex graphs in $F_e(3, 3; 4)$. They then used computations
to process all 153 such graphs in order to reconstruct all possible graphs $G$. However, all
graphs reconstructed did not arrow $(3, 3)$, showing $F_e(3, 3; 4) > 18$.

The long history of $F_e(3, 3; 4)$ is not only interesting in itself but also gives insight into
how difficult the problem is. Finding good bounds on the smallest order of any Folkman
graph (with fixed parameters) seems to be difficult, and some related Ramsey graph coloring
problems are NP-hard or lie even higher in the polynomial hierarchy. For example, Burr [6]
showed that arrowing $(3, 3)$ is coNP-complete, and Schaefer [76] showed that for general
graphs $F, G$, and $H$, $F \rightarrow (G, H)$ is $\Pi_2^P$-complete. The latter result is particularly significant,
as it provides a natural problem that is complete for a higher level of the polynomial hierarchy.
2.3 Arrowing and MAX-CUT

Building off Spencer’s and the other methods described above, Dudek and Rödl [19] in 2008 showed how to construct a graph $H_G$ from a graph $G$, such that the maximum size of a cut of $H_G$ determines whether or not $G \rightarrow (3, 3)$. They construct the graph $H_G$ as follows. The vertices of $H_G$ are the edges of $G$, so $|V(H_G)| = |E(G)|$. For $e_1, e_2 \in V(H_G)$, if edges \{e_1, e_2, e_3\} form a triangle in $G$, then \{e_1, e_2\} is an edge in $H_G$.

Let $t_\triangle(G)$ denote the number of triangles in graph $G$. Clearly, $|E(H_G)| = 3t_\triangle(G)$. Let $MC(H)$ denote the MAX-CUT size of graph $H$.

Theorem 4 (Dudek and Rödl [19]). $G \rightarrow (3, 3)$ if and only if $MC(H_G) < 2t_\triangle(G)$.

There is a clear intuition behind Theorem 4 that we will now describe. Any edge 2-coloring of $G$ corresponds to a bipartition of the vertices in $H_G$. If a triangle colored in $G$ is not monochromatic, then its three edges, which are vertices of $H_G$, will be separated in the bipartition. If we treat this bipartition as a cut, then the size of the cut will count each triangle twice for the two edges that cross it. Since there is only one triangle in a graph that contains two given edges, this effectively counts the number of non-monochromatic triangles. Therefore, if it is possible to find a cut that has size equal to $2t_\triangle(G)$, then such a cut defines an edge coloring of $G$ that has no monochromatic triangles. However, if $MC(H_G) < 2t_\triangle(G)$, then in each coloring, all three edges of some triangle are in one part and thus, $G \rightarrow (3, 3)$.

A benefit of converting the problem of arrowing $(3, 3)$ to MAX-CUT is that the latter is well-known and has been studied extensively in computer science and mathematics (see for example [16]). The decision problem MAX-CUT($H, k$) asks whether or not $MC(H) \geq k$. It is known that MAX-CUT is NP-hard and the decision version was one of Karp’s 21 NP-complete problems [46]. In our case, $G \rightarrow (3, 3)$ if and only if MAX-CUT($H_G, 2t_\triangle(G)$) doesn’t hold. Since MAX-CUT is NP-hard, an attempt is often made to approximate it, such as in the approaches presented in the next two sections.

2.3.1 Minimum Eigenvalue Method

A method exploiting the minimum eigenvalue was used by Dudek and Rödl [19] to show that some large graphs are members of $F_e(3, 3; 4)$. The following upper bound (2.1) on $MC(H_G)$ can be found in [19], where $\lambda_{\min}$ denotes the minimum eigenvalue of the adjacency matrix of $H_G$.

$$MC(H_G) \leq \frac{|E(H_G)|}{2} - \frac{\lambda_{\min}|V(H_G)|}{4}. \quad (2.1)$$

The proof of this bound is quite simple. Let $x = (x_1, x_2, \ldots, x_n)$ and $x_i \in \{-1, 1\}$ for all $1 \leq i \leq n$. For a cut $\{S, \overline{S}\}$ of $H_G = (V, E)$, let $x_i = 1$ if vertex $i$ is in $S$ and $x_i = -1$ if $i$ is in $\overline{S}$. Clearly, $\frac{1}{4} \sum_{\{i,j\} \in E} (x_i - x_j)^2$ counts the size of the cut. Let $A$ be the adjacency
matrix of $H_G$, where $a_{ij} = 1$ if $\{i, j\} \in E(H_G)$ and $a_{ij} = 0$ otherwise. Then,

$$
\sum_{\{i,j\} \in E} (x_i - x_j)^2 = \sum_{\{i,j\} \in E} x_i^2 + x_j^2 - 2x_i x_j
= \sum_{i=1}^{n} \text{deg}(i) \cdot x_i^2 - \sum_{\{i,j\} \in E} 2x_i x_j
= \sum_{i=1}^{n} \text{deg}(i) - \sum_{i,j} a_{ij} x_i x_j
= 2|E| - x^T A x.
$$

Because $A$ is symmetric, from the Rayleigh-Ritz ratio (see for example Theorem 4.2.2 in [40]), we know that $y^T A y \geq \lambda_{\min} \|y\|^2$ for all $y \in \mathbb{R}^n$. Then, $2|E| - x^T A x \leq 2|E| - \lambda_{\min} \|x\|^2$ and $\|x\|^2 = |V|$, giving the inequality in (2.1).

Dudek and Rödl used (2.1) to prove the following theorem:

**Theorem 5** (Dudek and Rödl [19]). $F_e(3,3;4) \leq 941$

**Proof.** For positive integers $r$ and $n$, if $-1$ is an $r$-th residue modulo $n$, then let $G(n,r)$ be a circulant graph on $n$ vertices with the vertex set $\mathbb{Z}_n$ and the edge set $E(G(n,r)) = \{\{u,v\} \mid u \neq v \text{ and } u - v \equiv \alpha \text{ mod } n, \text{ for some } \alpha \in \mathbb{Z}_n\}$.

The graph $G_{941} = G(941,5)$ has 707632 triangles. Using the MATLAB [58] eigs function, Dudek and Rödl [19] computed

$$
MC(H_{G_{941}}) \leq 1397484 < 1415264 = 2\Delta(G_{941}).
$$

Thus, by Theorem 1, $G_{941} \rightarrow (3,3)$.

In an attempt to improve $F_e(3,3;4) \leq 941$, we tried removing vertices of $G_{941}$ to see if the minimum eigenvalue bound would still show arrowing. We applied multiple strategies for removing vertices, including removing neighborhoods of vertices, randomly selected vertices, and independent sets of vertices. Most of these strategies were successful, and led to the following theorem:

**Theorem 6.** $F_e(3,3;4) \leq 860$.

**Proof.** For a graph $G$ with vertices $\mathbb{Z}_n$, define $C = C(d,k) = \{v \in V(G) \mid v = id \text{ mod } n, \text{ for } 0 \leq i < k\}$. Let $G = G_{941}$, $d = 2$, $k = 81$, and $G_C$ be the graph induced on $V(G) \setminus C(d,k)$. Then $G_C$ has 860 vertices, 73981 edges and 542514 triangles. Using the MATLAB eigs function, we obtain $\lambda_{\min} \approx -14.663012$. Setting $\lambda_{\min} > -14.664$ in (2.1) gives

$$
MC(H_{G_C}) < 1084985 < 1085028 = 2\Delta(G_C).
$$

Therefore, $G_C \rightarrow (3,3)$.
None of the methods used allowed for 82 or more vertices to be removed without the upper bound on $MC$ becoming larger than $2t_\Delta$.

**Small Examples**

Although the minimum eigenvalue method led to results, it is not consistent for showing arrowing with small examples. Let $\alpha$ be the upper bound of $MC(H_G)$ computed with this method and let $\beta = 2t_\Delta(G)$.

An example of when it is successful is $G = K_6$, the upper bound witness for $R(3,3) = 6$. We construct $H_{K_6}$ and obtain $|V(H_{K_6})| = 15$ and $|E(H_{K_6})| = 3t_\Delta(K_6) = 60$. We compute $\lambda_{\min}(H_{K_6}) = -2$ and

$$\alpha = \frac{60}{2} - \frac{(-2)(15)}{4} = 37.5, \quad \beta = 40.$$  

Since $\alpha < \beta$ the $\lambda_{\min}$ method successful shows that $K_6 \rightarrow (3,3)$.

However, the method fails for the next simplest case, $G = K_3 + C_5$. We construct $H_G$, with $|V(H)| = 23$ and $|E(H)| = 93$, and compute $\lambda_{\min}(H_G) \approx -3.3393$. Then,

$$\alpha = \frac{93}{2} - \frac{(-3.3393)(23)}{4} = 65.6993, \quad \beta = 62.$$  

Since $\alpha > \beta$, we cannot determine $K_3 + C_5 \rightarrow (3,3)$ using this method. The fact that (2.1) fails for this case was a main motivation for finding other methods which place upper bounds on the MAX-CUT of a graph. The next section discusses the Goemans-Williamson semi-definite programming MAX-CUT relaxation, which we successfully used to further improve the upper bound of $F_\epsilon(3,3;4)$.

**2.3.2 Goemans-Williamson Method**

The Goemans-Williamson MAX-CUT approximation algorithm [30] is a well-known, polynomial-time algorithm that relaxes the problem to a semi-definite program (SDP). It involves the first use of SDP in combinatorial approximation and has since inspired a variety of other successful algorithms (see for example [45, 26]). This randomized algorithm returns a cut with expected size at least 0.87856 of the optimal value. However, in our case, all that is needed is a feasible solution to the SDP, as it gives an upper bound on $MC(H)$. A brief description of the Goemans-Williamson relaxation follows.

The first step in relaxing MAX-CUT is to represent the problem as a quadratic integer program. Given a graph $H$ with $V(H) = \{1, \ldots, n\}$ and nonnegative weights $w_{i,j}$ for each pair of vertices $\{i,j\}$, we can write the MAX-CUT of $H$ as the following objective function:

Maximize $\frac{1}{2} \sum_{i<j} w_{i,j}(1 - y_i y_j)$ \hfill (2.3)

subject to: $y_i \in \{-1, 1\}$ for all $i \in V(H)$.  

Define one part of the cut as $S = \{ i \mid y_i = 1 \}$. Since in our case all graphs are weightless, we will use

$$w_{i,j} = \begin{cases} 1 & \text{if } \{i, j\} \in E(H), \\ 0 & \text{otherwise.} \end{cases}$$

Next, the integer program (2.3) is relaxed by extending the problem to higher dimensions. Each $y_i \in \{-1, 1\}$ is now replaced with a vector on the unit sphere $v_i \in \mathbb{R}^n$, as follows:

Maximize $\frac{1}{2} \sum_{i<j} w_{i,j} (1 - v_i \cdot v_j)$ \quad (2.4)

subject to: $\|v_i\| = 1$ for all $i \in V(H)$.

If we define a matrix $Y$ with the entries $y_{i,j} = v_i \cdot v_j$, that is, the Gram matrix of $v_1, \ldots, v_n$, then $y_{i,i} = 1$ and $Y$ is positive semi-definite. Therefore, (2.4) is a semidefinite program. We can write the SDP in the same form as (1.2).

Maximize $\frac{1}{2} \sum_{i<j} w_{i,j} (1 - y_{i,j})$ \quad (2.5)

subject to: $y_{i,i} = 1$ for all $i \in V(H)$,

$Y \succeq 0$.

Cholesky decomposition can then be performed on $Y$ to obtain the vectors $v_1, \ldots, v_n$. Once they are obtained, a simple rounding technique is used to obtain an approximate cut. The idea is to generate a random uniformly distributed vector $r$ and let $S^* = \{ i \mid v_i \cdot r \geq 0 \}$ be one part of the cut. The vector $r$ can be seen as the normal of a hyperplane that “cuts” the unit sphere, partitioning the vectors $v_1, \ldots, v_n$ which are on it. If $MC^*(H)$ is the size of the cut $(S^*, S^*^c)$, then some analysis yields $\mathbb{E}[MC^*(H)] \geq \alpha_{GW} MC(H)$, where $\mathbb{E}[MC^*(H)]$ is the expected value and $\alpha_{GW} > 0.87856$. However, as the actual maximum value of (2.5) is an upper bound on $MC(H)$, completing this last step is unnecessary for this work.

### 2.4 Experiments

Using the Minimum Eigenvalue and Goemans-Williamson approaches, we tested a wide variety of graphs for arrowing by finding upper bounds on MAX-CUT. These graphs included the $G(n, r)$ graphs tested by Dudek and Rödl, similar circulant graphs based on the Galois fields $GF(p^k)$, and different types of random graphs. Various modifications of these graphs were also considered, including the removal and/or addition of vertices and/or edges, as well as copying or joining multiple candidate graphs together in various ways. We detail such experiments in this section.

Multiple SDP solvers that were designed [5, 38] to handle large-scale SDP and MAX-CUT problems were used for the tests. Specifically, we made use of a version of SDPLR by Samuel Burer [5], a solver that uses low-rank factorization. The version SDPLR-MC includes
specialized code for the MAX-CUT SDP relaxation. SBmethod by Christoph Helmberg [38] implements a spectral bundle method and was also applied successfully in our experiments. In all cases where more than one solver was used, the same results were obtained.

Throughout this section, we use $\alpha$ to denote the computed upper bound of $MC(H_G)$ and $\beta$ to denote $2t_\Delta(G)$. We make use of the parameter $\rho = (\alpha - \beta)/\alpha$, as used by Dudek and Rödl [19], to estimate how “close” the methods are to showing $G \rightarrow (3, 3)$.

### 2.4.1 Graphs

We tested the graph $G_C$ of Theorem 6 with the SDP relaxation and obtained the upper bound $MC(H_{G_C}) \leq 1077834$, a significant improvement over the bound 1084985 obtained from the minimum eigenvalue method. This provides further evidence that $G_C \rightarrow (3, 3)$, and is an example of when (2.4) yields a much better upper bound.

The type of graph that led to the best results, including an improvement to the upper bound of $F_{\epsilon}(3, 3; 4)$, was described by Lu [56]. We discuss these graphs and our results in the next section.

$G_{127}$

Define graph $G_{127}$ as $V(G_{127}) = \mathbb{Z}_{127}$ and $E(G_{127}) = \{\{x, y\} \mid x - y \equiv \alpha^3 \mod 127\}$ (that is, the graph $G(127, 3)$ as defined in the Section 2.3.1). This graph was given particular attention throughout this work, as it has been conjectured by Exoo that $G_{127} \rightarrow (3, 3)$. He also suggested that subgraphs induced on less than 100 vertices of $G_{127}$ may as well, which would give a positive answer to Graham’s question of whether $F_{\epsilon}(3, 3; 4) < 100$.

$G_{127}$ has 2667 edges, 9779 triangles, is $K_4$-free, and has an independence number of 11. It is regular of degree 42 and is both vertex- and edge-transitive. The graph was originally defined by Hill and Irving in 1982 [39] and was used to show $R(4, 4, 4) \geq 128$, as the edges of $K_{127}$ can be three-colored in such a way that each color is isomorphic to it.

An upper bound of 20181 for $MC(H_{G_{127}})$ was obtained by both the $\lambda_{\min}$ and SDP methods. As $2t_\Delta(G_{127}) = 19558$, the approaches fail to show $G_{127} \rightarrow (3, 3)$. However, the “closeness” obtained is $\rho = 0.03088$, a relatively low value. Multiple attempts were made at modifying $G_{127}$ in order to lower $\rho$, including removing edges and vertices, and multiple copies of $G_{127}$ were attached together in a variety of ways. However, in every case, the modified graph had a $\rho$ value greater than 0.03088.

$G_{127}$ contains three disjoint independent sets of order 11. These sets were removed one-by-one and the resulting graphs were tested for arrowing. The results are presented in Table 2.3. Note that although $\rho$ increases for both methods, the SDP $\rho$ increases much less. This was a common trend amongst all experiments performed – the SDP bounds tended to be better, especially when a graph was less structured.

### Circulant Graphs

A number of circulant graphs not defined by residues were tested. One such graph $G_{199}$ was given particular attention, as it appears to be a viable candidate for arrowing $(3, 3)$. $G_{199}$ is
Table 2.3: MAX-CUT tests with \( G_{127} \) and its independent sets removed

| # Removed | \( |E(G)| \) | \( 2t_\Delta(G) \) | \( \lambda_{\text{min}} \) | \( \rho(\lambda_{\text{min}}) \) | SDP | \( \rho(\text{SDP}) \) |
|-----------|-----------|---------------|-------------|----------------|------|----------------|
| 0         | 2667      | 19558         | 20181       | 0.03088        | 20181| 0.03088        |
| 1         | 2205      | 14476         | 15285       | 0.05293        | 15073| 0.03961        |
| 2         | 1801      | 10670         | 11529       | 0.07451        | 11213| 0.04843        |
| 3         | 1455      | 7836          | 8617        | 0.09064        | 8307 | 0.05670        |

Defined as \( V(G_{199}) = Z_{199} \) and \( E(G_{199}) = \{ \{ u, v \} \mid u - v \in D \} \), where

\[
D = \{ 1, 2, 4, 13, 15, 19, 21, 24, 26, 27, 30, 33, 37, 38, 42, 43, 48, 51, 58, 74, 76, 83, 84, 86, 92, 93, 96 \}.
\]

\( G_{199} \) is 54-regular with 5373 edges and 21492 triangles, and does not contain a \( K_4 \). The \( \lambda_{\text{min}} \) method gave \( \alpha = 45497 \) and \( \rho = 0.05523 \), while the SDP method gave \( \alpha = 45173 \) and \( \rho = 0.04846 \). Although these tests failed to show \( G_{199} \rightarrow (3, 3) \), the \( \rho \) values are still relatively low, and it is still quite possible that \( G_{199} \in F_e(3, 3; 4) \).

### Additional Graphs

The \( G(n, r) \) graphs given by Dudek-Rödl were tested for all primes \( 100 \leq n \leq 941 \) and all results obtained agreed with theirs. Similar residue-based circulant graphs with prime-power orders, built over Galois fields, were also tested. Generating such graphs was accomplished with the Number Theory Library by Victor Shoup [78], a C++ library that includes data structures and algorithms for performing operations with polynomials over finite fields. Unfortunately, most graphs generated this way contained many \( K_4 \)'s, and those that did not performed poorly with the MAX-CUT tests.

Many different types of random graphs were tested. \( G(n, p) \) graphs with varying \( 50 \leq n \leq 1000 \) and \( p \) were made \( K_4 \)-free by removing a random edge from each \( K_4 \). \( K_4 \)-free graphs were also generated by randomly permuting all possible edges, and adding them via the random order when no \( K_4 \) was formed. Circulant graphs were generated in a similar way: for a graph on \( n \) vertices, the possible distances \( 1, 2, \ldots, \lfloor n/2 \rfloor \) were randomly permuted and the circulant edges were added in the random order when no \( K_4 \) was formed. No such graphs generated by any of these approaches were feasible Folkman candidates, and both MAX-CUT methods failed significantly, with \( \rho \) values often in the range \((0.1, 0.4)\). This possibly suggests that well-structured graphs such as \( G_{127} \) and \( G_{199} \) are more likely to arrow \((3, 3)\), and are better suited for such testing.

### 2.4.2 SAT-solvers

In addition to the MAX-CUT methods, testing of graphs was done using a reduction from arrowing triangles to the Boolean satisfiability problem, 3SAT. An instance of 3SAT consists of a Boolean formula in conjunctive normal form, that is, a conjunction of clauses where each clause is a disjunction of, in this case, three literals. The goal is to decide whether the
formula can be satisfied (evaluated to TRUE) by some assignment of the variables. The general SAT problem was the first known \textbf{NP}-complete problem as shown in the well-known Cook-Levin Theorem.

*** What do I cite for Cook-Levin?

Given graph \( G \), we can decide if \( G \to (3, 3) \) by deciding the satisfiability of the Boolean formula \( \phi(G) \), constructed as follows. For all \( e_1, e_2, e_3 \in E(G) \) such that \( \{e_1, e_2, e_3\} \) is a triangle, we add the clauses \((e_1 \lor e_2 \lor e_3)\) and \((\overline{e_1} \lor \overline{e_2} \lor \overline{e_3})\) to \( \phi(G) \). Then,

\[ G \not\rightarrow (3, 3) \text{ iff } \phi(G) \text{ is satisfiable.} \]

The assignments of TRUE and FALSE to the literals are equivalent to the assignments of red and blue to the edges. The pair of clauses corresponding to a triangle \( \{e_1, e_2, e_3\} \) evaluates to TRUE only when the triangle is non-monochromatic, as an edge assigned TRUE yields \((e_1 \lor e_2 \lor e_3)\) TRUE and an edge assigned FALSE yields \((\overline{e_1} \lor \overline{e_2} \lor \overline{e_3})\) TRUE. Thus, \( \phi(G) \) is satisfied only when every triangle is non-monochromatic.

Large 3SAT instances can often be solved using specialized software, most of which compete in the biennial international SAT competition [17]. A number of these SAT-solvers were used for additional testing of Folkman graph candidates. The software used included \texttt{clasp} [28], which won one silver and two gold medals in the “Crafted” 2009 competition, and \texttt{glucose} [2], which won a gold medal in the “Application” 2011 competition.

Unfortunately, the SAT-solvers were unable to determine any cases of arrowing which were not previously known, or determined by MAX-CUT.

*** Cite Chris Wood’s recent work with \( G_{127} \) and SAT?

2.5 \( F_e(3.3; 4) \leq 786 \)

For positive integers \( n \) and \( s < n \), \( s \) relatively prime to \( n \), define set \( S = \{s^i \mod n \mid i = 0, 1, \ldots, m - 1\} \), where \( m \) is the smallest positive integer such that \( s^m \equiv 1 \mod n \). If \(-1 \mod n \in S\), then let \( L(n, s) \) be a circulant graph on \( n \) vertices with \( V(L(n, s)) = \mathbb{Z}_n \). For vertices \( u \) and \( v \), \( \{u, v\} \) is an edge of \( L(n, s) \) if and only if \( u - v \in S \). Note that the condition that \(-1 \mod n \in S\) implies that if \( u - v \in S \) then \( v - u \in S \).

In Table 1 of [56], a set of potential members of \( F_e(3, 3; 4) \) of the form \( L(n, s) \) were listed, and the graph \( L(9697, 4) \) was shown to arrow \((3, 3)\). Lu gave credit to Exoo for showing that \( L(17, 2) \), \( L(61, 8) \), \( L(79, 12) \), \( L(421, 7) \), and \( L(631, 24) \) do not arrow \((3, 3)\).

We tested all graphs from Table 1 of [56] of order less than 941 with the MAX-CUT method, using both the minimum eigenvalue and SDP upper bounds. Table 2.4 lists the results. Note that although none of the computed upper bounds of the \( L(n, s) \) graphs imply arrowing \((3, 3)\), all SDP bounds match those of the minimum eigenvalue bound. This is distinct from other families of graphs, including those in [19], as the SDP bound is usually tighter. Thus, these graphs were given further consideration.

Numerous attempts were made at modifying these graphs in hopes that one of the MAX-CUT methods would be able to prove arrowing. \( L(127, 5) \) was given particular attention, as it is the same graph as \( G_{127} \) discussed in the previous section. Although we were unable to
Table 2.4: Potential $F_e(3; 3; 4)$ graphs $G$ and upper bounds on $MC(H_G)$, where “$\lambda_{\text{min}}$” is the bound (2.1) and “SDP” is the solution of (2.4) from SDPLR-MC and SBmethod. $G_{786}$ is the graph of Theorem 7.

<table>
<thead>
<tr>
<th>$G$</th>
<th>$2t_\triangle(G)$</th>
<th>$\lambda_{\text{min}}$</th>
<th>SDP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L(127, 5)$</td>
<td>19558</td>
<td>20181</td>
<td>20181</td>
</tr>
<tr>
<td>$L(457, 6)$</td>
<td>347320</td>
<td>358204</td>
<td>358204</td>
</tr>
<tr>
<td>$L(761, 3)$</td>
<td>694032</td>
<td>731858</td>
<td>731858</td>
</tr>
<tr>
<td>$L(785, 53)$</td>
<td>857220</td>
<td>857220</td>
<td>857220</td>
</tr>
<tr>
<td>$G_{786}$</td>
<td>857762</td>
<td>857843</td>
<td>857753</td>
</tr>
</tbody>
</table>

We note that finding a lower bound on MAX-CUT, such as the $857742 \leq MC(H_{G_{786}})$ bound from SpeeDP, follows from finding an actual cut of a certain size. This method may be useful, as finding a cut of size $2t_\triangle(G)$ shows that $G \not\rightarrow (3; 3)$.
method. This is especially apparent when $H$ has less symmetrical structure. However, both methods appear insufficient for further improvements to the upper bound. They both fail to show arrowing for easy cases, such as all 659 15-vertex graphs in $F_e(3,3;5)$ and some other small cases presented in Table 2.5. A possible reason for these failures is that the small order of the graphs leaves little room for the error inherent in approximations, suggesting that the approximations work well when the graphs are larger. This seems to create a gap, where graphs of interest such as $G_{127}$ are too small for approximation methods like SDP-solvers but are too large for exact methods like SAT-solvers.

It is therefore likely that a new method is needed for further improvements. A possible strategy is to attempt the computation of the exact solution of the MAX-CUT IP (2.3) via approaches like Rendl, Rinaldi, and Wiegele’s SDP based branch & bound algorithm [74] used in their Biq Mac software [73]. Another possible thread of work is to try to prove $\phi(G)$ is unsatisfiable using methods other than exact SAT-solvers. For example, computing an upper bound on the maximum number of satisfiable clauses can potentially show unsatisfiability. Approximation algorithms for MAX-SAT, such as Karloff and Zwick’s SDP based algorithm [45] and Maaren, Norden, and Heule’s sums of squares based algorithm [81], may be worth investigation.

Another open question is the lower bound on $F_e(3,3;4)$, as it is quite puzzling that only 19 is the best known. Even an improvement to $20 \leq F_e(3,3;4)$ would be good progress.