On Derivatives

A CSCI-633 Mini Crash-Course

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Why Do We Need Derivatives?

In life, things are constantly "changing." Often, we are interested in how physical quantities change.

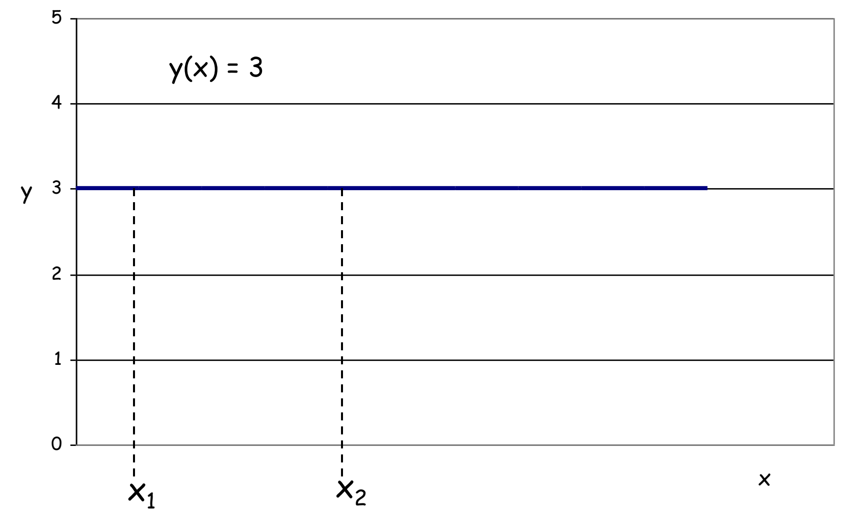
For example, how does an object's velocity change over time? How does the force acting on an object change over a distance traveled?
Such changes are described mathematically by *derivatives*

- *A* "derivative" = fancy name that describes how something changes with respect to (w.r.t.) something else

Getting Started

Let's start simple. Consider the function $y(x) = 3$ shown in the figure to the right. If you were asked "*How does this function change with x?*" or equivalently, "*How does y change as a function of x?*," you would say, "*It doesn't change. It is a constant value of 3 everywhere.*"

How would we describe your response mathematically?



Mathematics of "Change"

Well, we define *the change in the function y(x) with respect to the variable x, Δy/Δx,* to be

$$\frac{\Delta y}{\Delta x} = \frac{y(x_2) - y(x_1)}{x_2 - x_1}.$$

In English, this equation will tell us how the function *y(x)* changes over the distance between x_1 and x_2. Since *y* always equals 3 in our previous example, then

$$\frac{\Delta y}{\Delta x} = \frac{3-3}{x_2 - x_1} = 0.$$

Therefore, the function *y(x)* does not change over the interval x_1 and x_2. This mathematics will be how we describe the change in any function with respect to a given variable.

Understanding the symbols…

The symbol "Δ," called *delta*, represents the *change* in a variable. Translated, Δy reads: "The change in y," which equals:
"y final minus y initial"
or mathematically:
"$\Delta y = y_{final} - y_{initial}$."

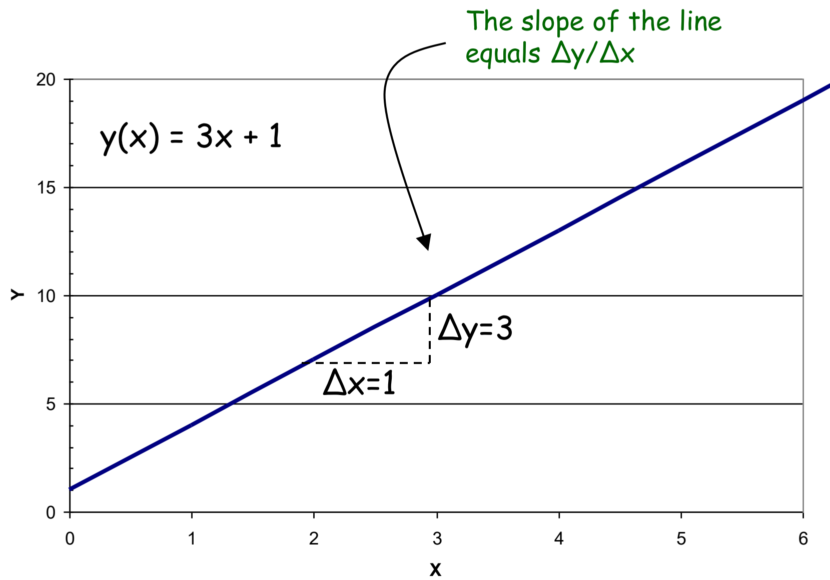
Example: a Straight Line

Now consider the function *y(x)* = 3*x* + 1 as drawn in the figure to the right. Again, how would *y(x)* change with *x*? Let's look at the interval between *x* = 2 and *x* = 3:

Using our definition for the change in *y(x)* with respect to *x* from the previous slide (hit the left arrow key if you need to back to the previous slide), we get:

$$\frac{\Delta y}{\Delta x} = \frac{y(x=3)-y(x=2)}{3-2} = \frac{[3*(3)+1]-[3*(2)+1]}{3-2} = \frac{10-7}{1} = 3$$

If we look at this graphically in Figure 2, we see that it is just the slope of the line!!! If we look at any interval of *x*, we would find that *y(x)* would change by the same amount, 3, over that interval because this function is just a straight line! Try it and see!!!



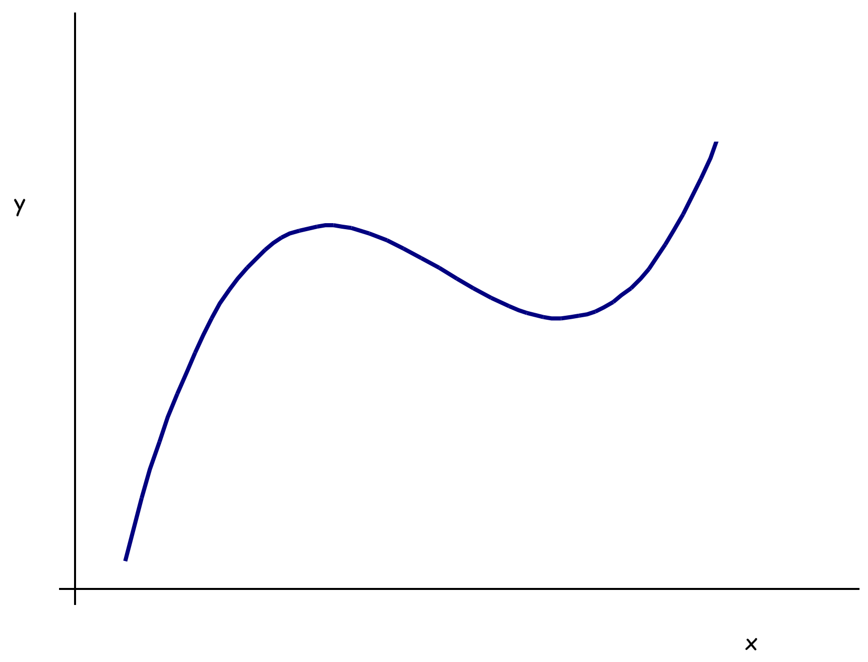
Δy/Δx = 3
The function y changes by 3 units between x=2 and x=3.

More than Straight Lines

On the previous slide, you learned that the slope of a straight line gives you the change in the function *y(x)* over the change in *x*. That concept will become the building block for everything that follows.

Now let us consider a more complicated function other than a straight line. Consider the graph to the right. How would you describe how the function *y(x)* changes at any given value of *x*?

Well, from what you just learned, the change in the function *y(x)* with respect to *x* should be given by the slope of a straight line. But what is the straight line???

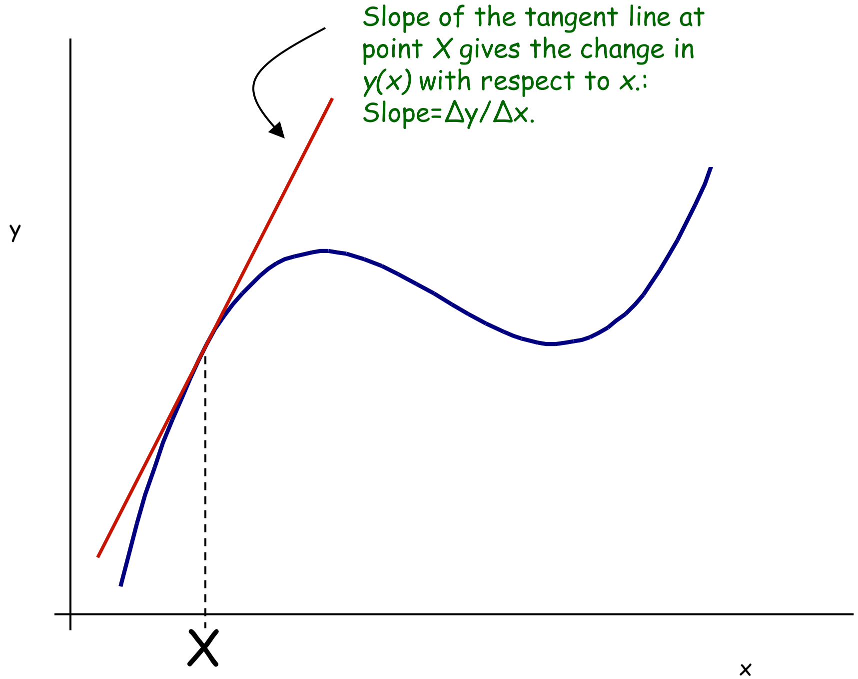


Graphs of Curves

Let us just get started and choose a point *X* on the graph. The question to be answered is "How does *y(x)* change at the point *X*?

To give the answer away, the change in *y(x)* with respect to *x* at point *X* is given by the slope of the **tangent line** at point *X* !!!

The question now is:
"How do we determine the slope of the tangent line at point *X*?



Understanding the terminology…

The word "tangent line" describes a line that intersects or touches a curve at only one point. At any given point on a smooth curve, there is only one unique tangent line and therefore there is only one value for the slope of that tangent line at that point.

Determining the Slope of the Tangent Line

To determine the slope of the tangent line, let us draw a different line that intersects the curve at both point *X* and point *X* + *h*.

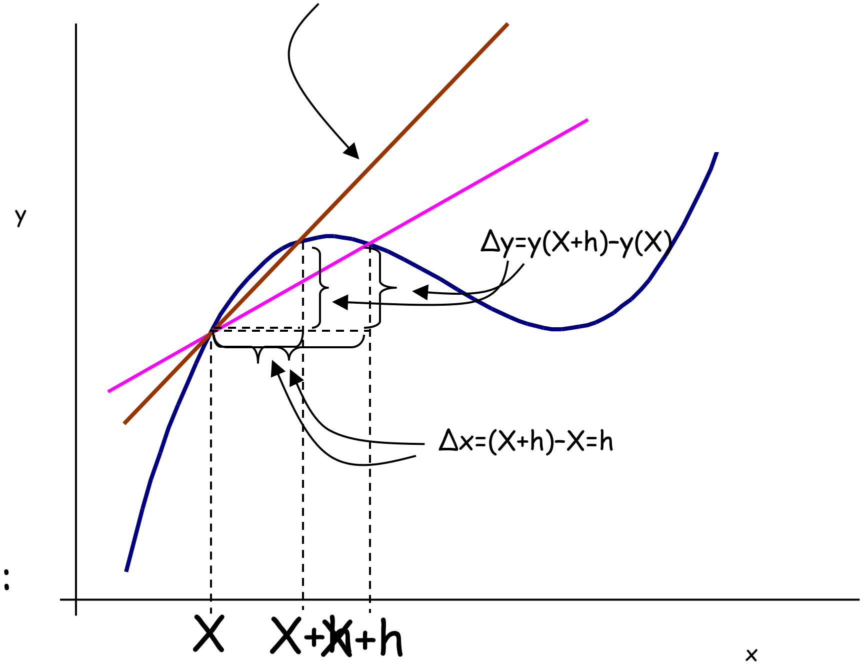
What is the slope of this line? From before, the slope of the line will be Δy/Δx:

$$\frac{\Delta y}{\Delta x} = \frac{y(X+h) - y(X)}{(X+h) - X} = \frac{y(X+h) - y(X)}{h}$$

This line is still not the tangent line at point *X*, but we can make it look more like the tangent line if we make the *h* a smaller value:

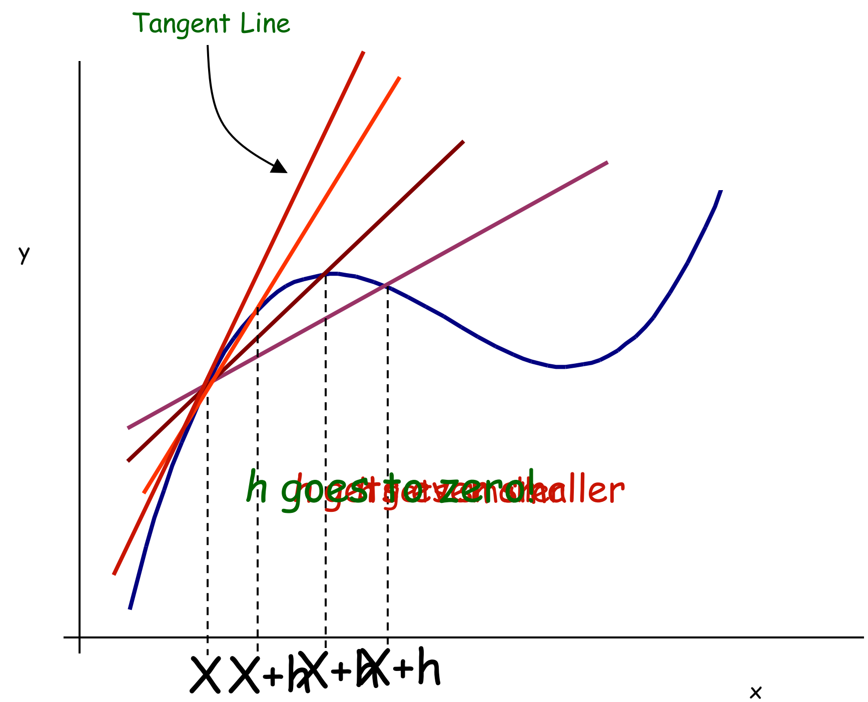
If you calculate the slope for this second line, it will have the same form as the above equation, except now *h* is a smaller value and therefore *y(X+h)* will be a different value.

Because *h* is smaller, the point *X+h* is closer to the original point *X* than before. Therefore, the slope of this second line is closer in value to the slope of the tangent line than what the slope of the first line was.



So What is a Derivative Anyway?

Now, if we keep making *h* smaller and smaller, then the line that passes through the points *X* and *X+h* will start looking more and more like the line tangent to the curve at point *X*. Eventually, as *h* goes to zero, then the line that goes through *X* and *X+h* **will become** the tangent line!!!



So, let's write this out in mathematics…

Slope of tangent line = $\frac{y(X+h)-y(X)}{h}$ (as *h* goes to zero)

We give the symbol $\frac{dy}{dx}$ to represent the slope of the tangent line.

This symbol, $\frac{dy}{dx}$, is what we call the ***derivative*** of *y* with respect to *x*.

Therefore, the term "derivative" just represents how the function *y(x) instantaneously* changes with respect to the variable *x*. As *h* goes to zero, $\Delta y/\Delta x$ becomes dy/dx.

Recap So Far

Let us recap what you have learned so far:

- The slope of a straight line tells you how the function *y(x)* changes as the variable *x* changes:

$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{y(x_{final}) - y(x_{initial})}{x_{final} - x_{initial}}$$

- The "derivative" of *y(x)* at point *X* is the slope of the tangent line to the curve of *y(x)* at point *X*.

- The derivative, *dy/dx*, is defined mathematically by the following equation:

$$\frac{dy}{dx} = \frac{y(x+h) - y(x)}{h} \quad \text{(as } h \text{ goes to zero)}$$

- The derivative, *dy/dx*, is the *instantaneous* change of the function *y(x)*.

- *As h goes to zero*, $\Delta y/\Delta x$ becomes dy/dx.

Using the Definition for Derivatives

Let us now apply our newly derived formula to calculate the derivative of $y(x) = x^2$.

$$\frac{dy}{dx} = \frac{y(x+h) - y(x)}{h} \quad \text{(as } h \text{ goes to zero)}$$

$$= \frac{((x+h)^2) - (x^2)}{h} \quad \text{(as } h \text{ goes to zero)}$$

$$= \frac{(x^2 + 2xh + h^2) - x^2}{h} \quad \text{(as } h \text{ goes to zero)}$$

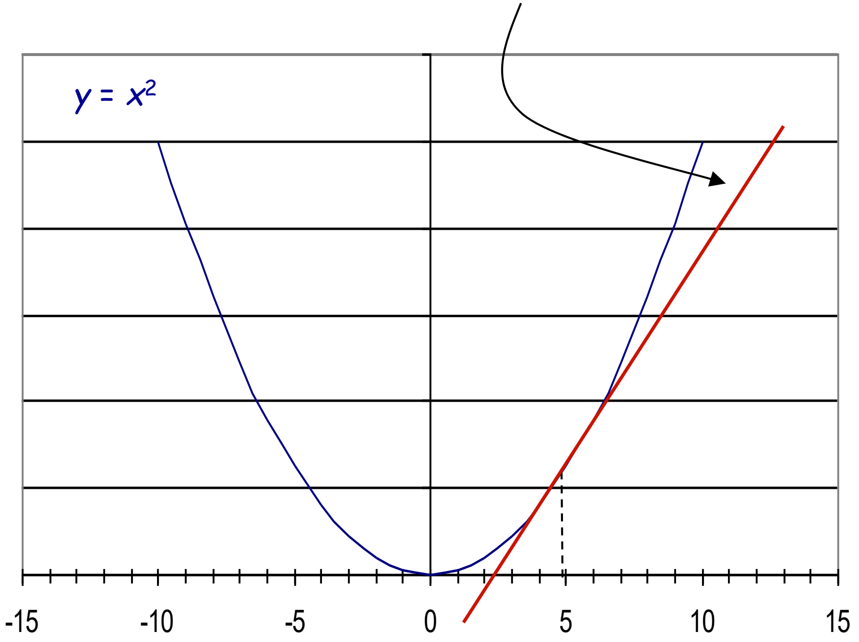
$$= \frac{2xh + h^2}{h} \quad \text{(as } h \text{ goes to zero)}$$

$$= 2x + h \quad \text{(as } h \text{ goes to zero)}$$

$$= 2x$$

And therefore, $\boxed{\frac{dy}{dx} = 2x}$

Tangent line at $x = 5$ has a slope of 10. Therefore, the function y(x) has an instantaneous slope of 10 units at x=5.



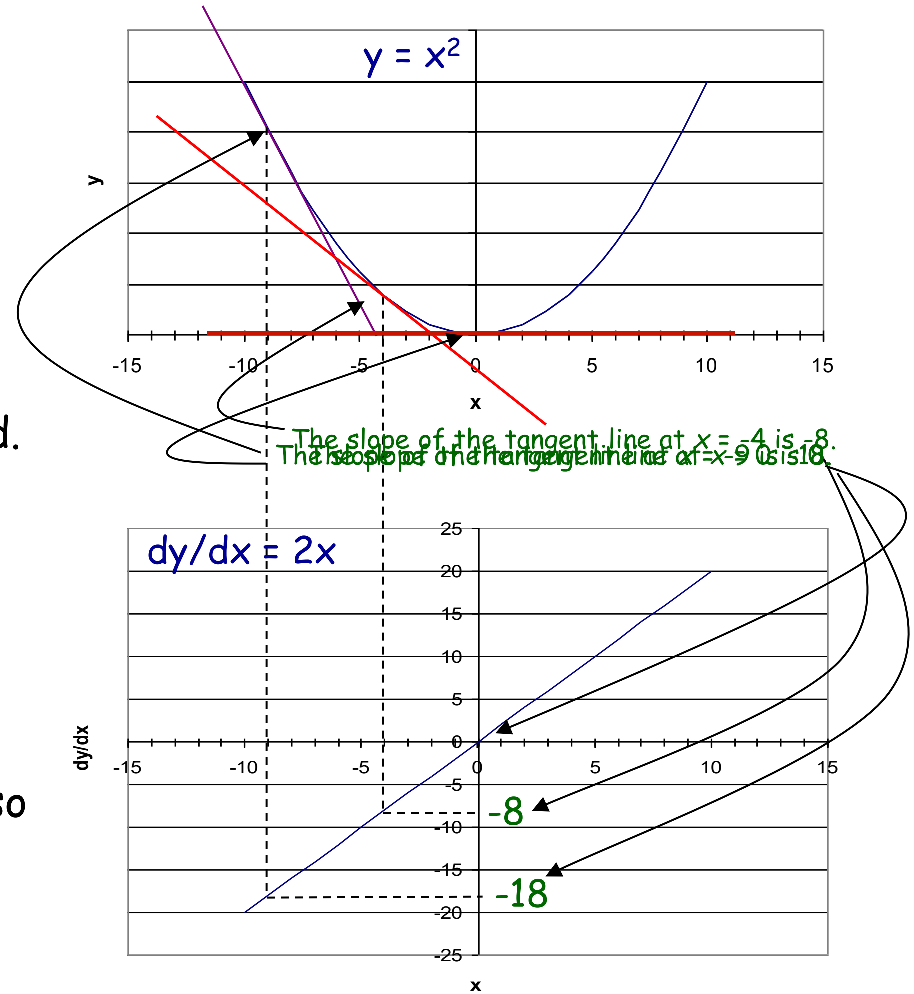
Let us use this result to determine the derivative at $x = 5$. Since the derivative of $y(x)=x^2$ equals $2x$, then the derivative at $x = 5$ is 2*5 = 10. Therefore, the slope of the tangent line that passes through $x = 5$ has a slope of 10!

Graphing the Derivative

In our previous example, we used the definition for the derivative to find the derivative of the function $y = x^2$. When we did this, we found the derivative to be a function itself: $dy/dx = 2x$. This is just a straight line as plotted to the bottom right.

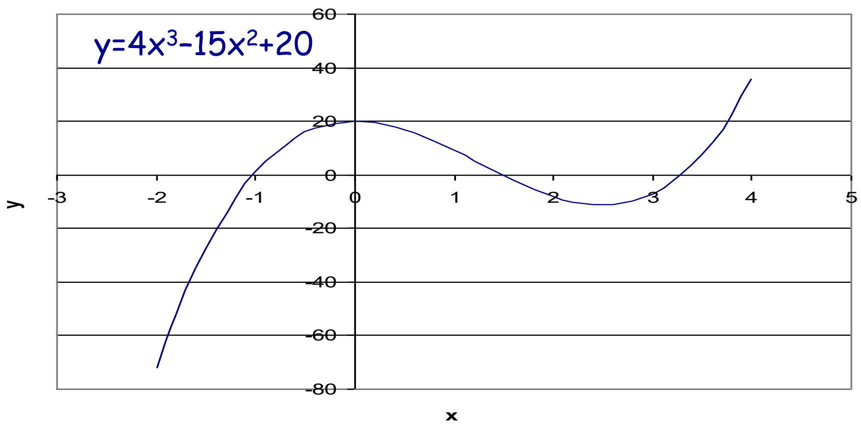
Let us see how the two graphs are related.

- You know that the derivative of a function is just the slope of that function. For example, look at the graph of $y = x^2$, for negative values of x, the slope of the tangent line should be negative. Looking at the graph of dy/dx, when x is negative, dy/dx is also negative!
- When the slope of the tangent line equals zero, then the value of the derivative will equal zero!



Another Example

Consider the formula $y = 4x^3 - 15x^2 + 20$.
This function is graphed to the right.



Calculating the derivative, we find:

$$\frac{dy}{dx} = \frac{y(x+h)-y(x)}{h} \quad \text{(as } h \text{ goes to zero)}$$ Definition of derivative

$$= \frac{[4(x+h)^3 - 15(x+h)^2 + 20] - [4x^3 - 15x^2 + 20]}{h} \quad \text{(as } h \text{ goes to zero)}$$ Substituted in the expression for $y(x)$

$$= \frac{[4(x^3 + 3x^2h + 3xh^2 + h^3) - 15(x^2 + 2xh + h^2) + 20] - [4x^3 - 15x^2 + 20]}{h} \quad \text{(as } h \text{ goes to zero)}$$

$$= \frac{12x^2h + 12xh^2 + h^3 - 30xh - 15h^2}{h} \quad \text{(as } h \text{ goes to zero)}$$ Terms that survived after some terms canceled

$$= 12x^2 + 12xh + h^2 - 30x - 15h \quad \text{(as } h \text{ goes to zero)}$$ Divided each term by h

$$\frac{dy}{dx} = 12x^2 - 30x$$ These terms survived after h went to zero

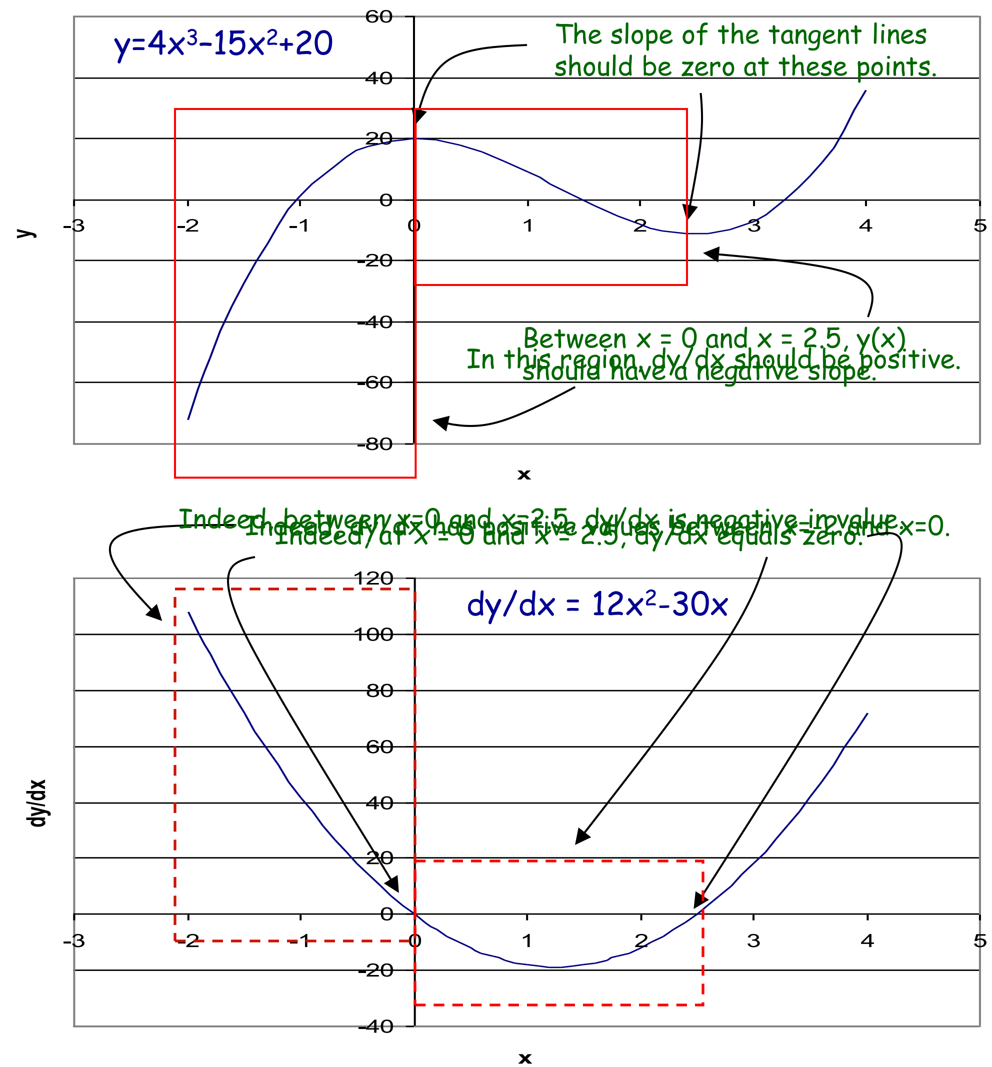
Example Continued

We have found that the derivative of $y(x) = 4x^3 - 15x^2 + 20$ to be:

$dy/dx = 12x^2 - 30x$. The graph of the original function is plotted in the top right while the graph of its derivative is plotted in the bottom right.

Let us compare the two graphs and some of their features:

- The original function *y(x) in* the region between $x = -2$ and $x = 0$ should have a positive slope.
- At $x = 0$ and at $x = 2.5$, *y(x)* has critical points (points where the slope of the tangent line equals zero) and therefore its derivative should equal zero at those points.
- Between $x = 0$ and 2.5, *y(x)* is decreasing in value which implies that its derivative is negative in this region.



The Shortcut (!)

You have seen so far two examples on calculating derivatives and their graphical representation and meaning. Certainly, the definition for the derivative can be used each time when one needs to be determined, but there exists a shortcut when it comes to functions of the form: $y(x) = Ax^n$, where "A" is just a numerical constant and "n" is an integer, positive or negative. Plugging this expression into the definition for the derivative, you will find that:

$$dy/dx = nAx^{n-1}$$

Using this shortcut to calculate the derivative of $y(x) = x^2$, we get:

$$dy/dx = 2*x^{2-1} = 2x.$$

This is exactly what we got when we used the definition of the derivative several slides ago. In our 2nd example, we found that the derivative of $y = 4x^3 - 15x^2 + 20$ to be:

$$dy/dx = 12x^2 - 30x$$

Does this shortcut work here? The answer = YES!!!

Some Simple Rules of Differentiation

The subject of derivatives is a huge branch of mathematics in of itself and cannot possibly be contained here in this small lecture series. Hopefully, though, you now have some knowledge and appreciation for what derivatives are. The following are commonly known formulas for derivatives.

1) $\frac{d}{dx}[c] = 0$

2) $\frac{d}{dx}[x] = 1$

3) $\frac{d}{dx}[cx^n] = ncx^{n-1}$

4) $\frac{d}{dx}[cf] = cf'$

5) $\frac{d}{dx}[f \pm g] = f' \pm g'$

6) $\frac{d}{dx}[f^n] = nf^{n-1}f'$

7) $\frac{d}{dx}[f * g] = f * g' + g * f'$

8) $\frac{d}{dx}[\sin x] = \cos x$

9) $\frac{d}{dx}[\cos x] = -\sin x$

10) $\frac{d}{dx}[\tan x] = \sec^2 x$

What are these symbols?

"f" and "g" are functions of x: f(x) and g(x).

"c" represents a constant numerical value and therefore is not a function of x.

"n" represents an integer number, positive or negative.

f' is shorthand for df/dx. Likewise, g' is shorthand for dg/dx.

Questions?