Some Ramsey Problems - Computational Approach

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Outline - Triangles Everywhere

or avoiding K_3 in some/most colors

1. Ramsey Numbers - Two Colors
   - Some known and computed facts
   - $R(3, 10)$ is hard
   - Some things to do, computationally
2. Ramsey Numbers - More Colors
   - Some general bounds
   - $R(3, 3, 4)$, $R(3, 3, 3, 3)$ are hard
   - Things to do
3. Most Wanted Folkman Number
   - Edge-arrowing $(3, 3)$
   - K_4-free edge-arrowing $(3, 3)$
   - Things to do
4. So, what to do next?

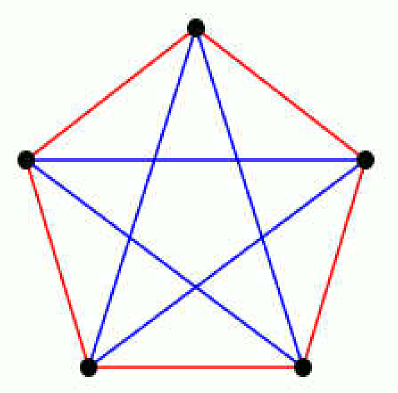
Ramsey Numbers

- $R(G, H) = n$ iff
  n = least positive integer such that in any 2-coloring of the edges of K_n there is a monochromatic G in the first color or a monochromatic H in the second color

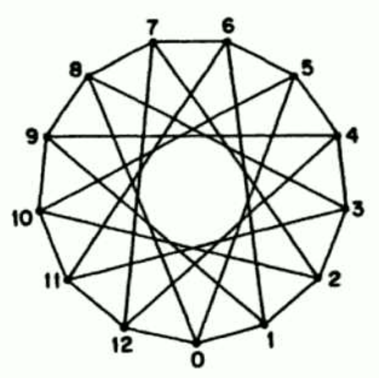
- $R(k, l) = R(K_k, K_l)$
- generalizes to r colors, $R(G_1, \cdots, G_r)$
- 2-*edge-colorings* $\cong$ *graphs*

- Theorem (Ramsey 1930): Ramsey numbers exist

Unavoidable classics



$R(3, 3) = 6$



$R(3, 5) = 14$ [GRS'90]

Basic upper bounds

- $R(k, l) = R(l, k), \quad R(k, 2) = k$
- Erdős, Szekeres 1935
  Greenwood, Gleason 1955

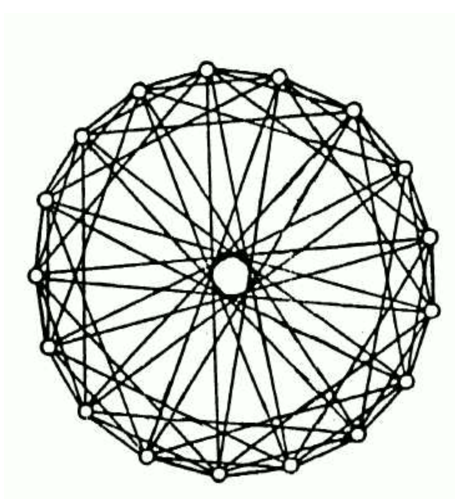
  $$R(k, l) \leq R(k-1, l) + R(k, l-1)$$

  with $<$ if both RHS terms are even, and

  $$R(k+1, l+1) \leq \binom{k+l}{k}$$

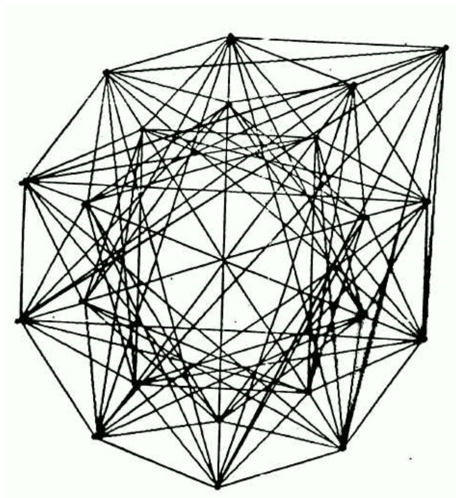
- $R(3,3) = 6, \quad R(3,4) = 9, \quad R(3,5) = 14, \quad R(4,4) = 18$

Unavoidable classics



$R(4,4) > 17, \quad dist = \{1, 2, 4, 8\}$

A messy case



$R(K_5 - e, K_5 - e) > 21$, double ring + outlier

Diagonal Cases

asymptotics

- Bounds (Erdős 1947, Spencer 1975, Thomason 1988)

$$\frac{\sqrt{2}}{e} 2^{n/2} n < R(n,n) < \binom{2n-2}{n-1} n^{-1/2+c/\sqrt{\log n}}$$

- Newest upper bound (Conlon, 2010)

$$R(n+1,n+1) \leq \binom{2n}{n} n^{-c\frac{\log n}{\log\log n}}$$

- **Conjecture** (Erdős 1947, \$100)
  $\lim_{n\to\infty} R(n,n)^{1/n}$ exists.
  If it exists, it is between $\sqrt{2}$ and 4 (\$250 for value).

Diagonal Cases

concretely

- Best construction (Frankl, Wilson 1981)

$$n^{c \log n / \log \log n} < R(n, n)$$

- First open case (Exoo 1989, MR 1997)

$$43 \leq R(5, 5) \leq 49$$

- Second open case (Kalbfleisch 1965, Mackey 1994)

$$102 \leq R(6, 6) \leq 165$$

Off-Diagonal Cases

asymptotics

- $R(3, *)$ - discussed later in the talk
- Bounds (Spencer 1977, Li and Rousseau 2000)

$$c_k \left(\frac{n}{\log n} \right)^{(k+1)/2} < R(k, n) < (1 + o(1)) \frac{n^{k-1}}{\log^{k-2} n}$$

- A generalization (Krivelevich 1995)
  $\rho(H) =$ largest density $(e-1)/(v-2)$ of subgraphs of H
  $\rho(K_k) = (k+1)/2$

$$c_H \left(\frac{n}{\log n} \right)^{\rho(H)} < R(H, n)$$

Off-Diagonal Cases

fixing small k

- $R(3, *)$ - later in these slides
- McKay-R 1995, $R(4, 5) = 25$
- Bohman triangle-free process - 2009

$$R(4, n) = \Omega(n^{5/2} / \log^2 n)$$

- Kostochka, Pudlák, Rődl - 2010
  constructive lower bounds

$$R(4, n) = \Omega(n^{8/5}), \quad R(5, n) = \Omega(n^{5/3}), \quad R(6, n) = \Omega(n^2)$$

(vs. probabilistic $5/2, 6/2, 7/2$ with /logs)

Values and Bounds on $R(k, l)$

two colors, avoiding cliques

k \ l	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 43	46 51	52 59	59 69	66 78	73 88
4		18	25	35 41	49 61	56 84	73 115	92 149	97 191	128 238	133 291	141 349	153 417
5			43 49	58 87	80 143	101 216	125 316	143 442	159	185 848	209	235 1461	265
6				102 165	113 298	127 495	169 780	179 1171	253	262 2566	317	5033	401
7					205 540	216 1031	233 1713	289 2826	405 4553	416 6954	511 10581	15263	22116
8						282 1870	317 3583	6090	10630	16944	817 27490	41525	861 63620
9							565 6588	580 12677	22325	39025	64871	89203	
10								798 23556		81200			1265

[ElJC survey *Small Ramsey Numbers, revision #12, 2009*]

General lower bound constructions

aren't that good

Theorem Burr, Erdős, Faudree, Schelp, 1989
$R(k,n) \geq R(k,n-1) + 2k - 3$ for $k \geq 2$, $n \geq 3$ (not $n \geq 2$)

Theorem (Xu-Xie-Shao-R 2004, 2010)
If $2 \leq p \leq q$ *and* $3 \leq k$, *then* $R(k, p+q-1) \geq$

$$R(k,p) + R(k,q) + \begin{cases} k-3, & \textit{if} \quad 2 = p \\ k-2, & \textit{if} \quad 3 \leq p \textit{ or } 5 \leq k \\ p-2, & \textit{if} \quad 2 = p \textit{ or } 3 = k \\ p-1, & \textit{if} \quad 3 \leq p \textit{ and } 4 \leq k \end{cases}$$

For $p = 2$, $n = q + 1$, we have $R(k,p) = k$,
which implies BEFR'89

Proof by construction

2004 cases

Given
(k,p)-graph G, (k,q)-graph H, $k \geq 3$, $p,q \geq 2$
G and H contain induced K_{k-1}-free graph M

construct
$(k, p+q-1)$-graph F, $n(F) = n(G) + n(H) + n(M)$

$VG = \{v_1, v_2, ..., v_{n_1}\}$, $VH = \{u_1, u_2, ..., u_{n_2}\}$
$VM = \{w_1, ..., w_m\}$, $m \leq n_1, n_2$, $K_{k-1} \not\subset M$
$G[\{v_1, ..., v_m\}]$, $H[\{u_1, ..., u_m\}] \cong M$
$\phi(w_i) = v_i$, $\psi(w_i) = u_i$ isomorphisms

$VF = VG \cup VH \cup VM$
$E(G,H) = \{\{v_i, u_i\} \mid 1 \leq i \leq m\}$
$E(G,M) = \{\{v_i, w_j\} \mid 1 \leq i \leq n_1, 1 \leq j \leq m, \{v_i, v_j\} \in E(G)\}$
$E(H,M) = \{\{u_i, w_j\} \mid 1 \leq i \leq n_2, 1 \leq j \leq m, \{u_i, u_j\} \in E(H)\}$

#vertices / #graphs

no exhaustive searches beyond 13

4	11
5	34
6	156
7	1044
8	12346
9	274668
10	12005168
11	1018997864
12	165091172592 $\approx 1.6 * 10^{11}$
too many to process	
13	50502031367952 $\approx 5 * 10^{13}$
14	29054155657235488
15	31426485969804308768
16	64001015704527557894928
17	245935864153532932683719776
18	$\approx 2 * 10^{30}$

#vertices / #triangle-free graphs

no exhaustive searches beyond 17

4	7
5	14
6	38
7	107
8	410
9	1897
10	12172
11	105071
12	1262180
13	20797002
14	467871369
15	14232552452
16	581460254001 $\approx 6 * 10^{11}$
too many to process	
17	$\approx 3 * 10^{12}$

Asymptotics

Ramsey numbers avoiding K_3

- Recursive construction yielding
  $R(3, 4k+1) \geq 6R(3, k+1) - 5$
  $\Omega(k^{\log 6/\log 4}) = \Omega(k^{1.29})$
  Chung-Cleve-Dagum 1993
- Explicit $\Omega(k^{3/2})$ construction
  Alon 1994, Codenotti-Pudlák-Giovanni 2000
- Kim 1995, lower bound
  Ajtai-Komlós-Szemerédi 1980, upper bound
  Bohman 2009, triangle-free process

$$R(3, k) = \Theta\left(\frac{k^2}{\log k}\right)$$

Small $R(3, k)$ cases

k	$R(3,k)$	year	reference [lower/upper]
3	6	1953	Putnam Competition
4	9	1955	Greenwood-Gleason
5	14	1955	Greenwood-Gleason
6	18	1964	Kéry
7	23	1966 / 1968	Kalbfleisch / Graver-Yackel
8	28	1982 / 1992	Grinstead-Roberts / McKay-Zhang
9	36	1966 / 1982	Kalbfleisch / Grinstead-Roberts
10	40-43	1989 / 1988	Exoo / Kreher-R

Known values of $R(3, k)$

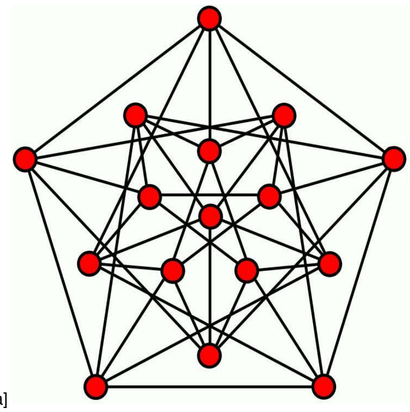
Erdős and Sós, 1980, asked about
$3 \leq \Delta_k = R(3, k) - R(3, k-1) \leq k$:

$$\Delta_k \xrightarrow{k} \infty \,? \qquad \Delta_k/k \xrightarrow{k} 0 \,?$$

Clebsch $(3, 6; 16)$-graph on $GF(2^4)$

$(x, y) \in E$ iff $x - y = \alpha^3$

[Wikipedia]



Alfred Clebsch (1833-1872)

Larger Cases

K_3 versus $K_k - e$ or K_k

$R(3, K_7 - e) = 21$ $\quad$ $R(3, K_8 - e) = 25$ $\quad$ $R(3, K_9 - e) = 31$

$R(3, 7) = 23$ $\quad$ $R(3, 8) = 28$ $\quad$ $R(3, 9) = 36$

All $R(3, K_k - e)$ critical graphs are known for $k \leq 8$

All $R(3, K_k)$ critical graphs are known for $k \leq 7$

First open cases:

$37 \leq R(K_3, K_{10} - e) \leq 38$, $42 \leq R(K_3, K_{11} - e) \leq 47$

$40 \leq R(K_3, K_{10}) \leq 43$, $46 \leq R(K_3, K_{11}) \leq 51$

Upper bounds by counting edges

computing $R(3, 10)$ is difficult

Definition: $e(k, n)$ = min # edges in n-vertex triangle-free graphs without independent sets of order k

- Very good lower bounds on $e(k-1, n-d)$ give good lower bounds on $e(k, n)$
- For any graph $G \in R(k, n, e)$

$$ne - \sum_{i=0}^{k-1} n_i(e(k-1, n-i-1) + i^2) \geq 0$$

- $e(9, n)$ not known for $27 \leq n \leq 35$ seem needed before improving on $e(10, n)$ for $n > 37$
- known $e(8, n)$-graphs not sufficient to improve on $e(9, n)$

$R(K_3, G)$

general non-asymptotic results

- $R(K_3, W_n) = R(C_3, C_n) = 2n - 1$
  Faudree-Schelp 1974, Burr-Erdős 1983
  all critical colorings, R-Jin 1994

- $R(K_3, G) = 2n(G) - 1$, for connected G
  $e(G) \leq 17(n(G) + 1)/15$, $n(G) \geq 4$
  Burr-Erdős-Faudree-Rousseau-Schelp 1980

- $R(K_3, G) \leq 2e(G) + 1$, isolate-free G
  $R(K_3, G) \leq n(G) + e(G)$, a conjecture for all G
  Sidorenko 1992-3, Goddard-Kleitman 1994

$R(K_3, G)$

general non-asymptotic results

- $R(nK_3, mK_3) = 2n + 3m$ for $n \geq m \geq 1, n \geq 2$
  Burr-Erdős-Spencer 1975
- $R(K_3, K_3 + \overline{K_n}) = R(K_3, K_3 + C_n) = 2n + 5$ for $n \geq 212$
  Zhou 1993
- $R(K_3, K_2 + T_n) = 2n + 3$ for $n \geq 4$
  Song-Gu-Qian 2004
- $R(K_3, G)$ for all connected G, $n(G) \leq 9$
  Brandt-Brinkmann-Harmuth 1998-2000

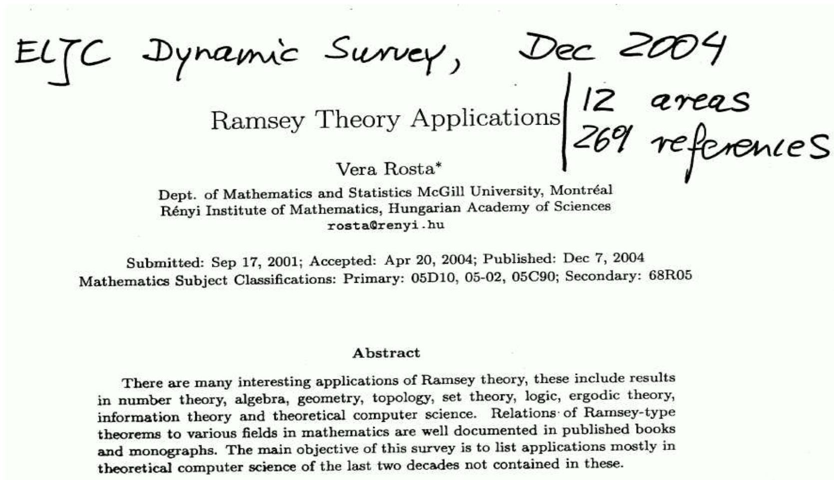
Things to do for two colors

avoiding triangles

- Enumerate all critical $(3, 8; 27)$-graphs
  430K+ known already
- Enumerate all critical $(3, 9; 35)$-graphs
  only one is known!
- Finish off $37 \leq R(3, K_{10} - e) \leq 38$
- $R(3, 10) \leq 43$, get it down first to 42

  $R(3, 10) \geq 40$, don't even try to do better :-(lower bound 40 is probably correct

Stay awake - applications exist



ELJC Dynamic Survey, Dec 2004

Ramsey Theory Applications

12 areas
269 references

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Abstract

There are many interesting applications of Ramsey theory, these include results in number theory, algebra, geometry, topology, set theory, logic, ergodic theory, information theory and theoretical computer science. Relations of Ramsey-type theorems to various fields in mathematics are well documented in published books and monographs. The main objective of this survey is to list applications mostly in theoretical computer science of the last two decades not contained in these.

More colors

upper bound

$$R(k_1, \ldots, k_r) \leq 2 - r + \sum_{i=1}^{r} R(k_1, \ldots, k_{i-1}, k_i - 1, k_{i+1}, \ldots, k_r)$$

with strict $<$ if the RHS is even and sum has en even term
Greenwood-Gleason 1955

Only two known multicolor cases, (3,3,4) and (3,3,3,3), where the RHS is improved. Likely this bound is never tight, except for (3,3,3).

More colors

some constructive results

- Xu-Xie-Exoo-R 2004
  - for $k_1 \geq 5$ and $k_i \geq 2$
    $R(k_1, 2k_2 - 1, k_3, \cdots, k_r) \geq 4R(k_1 - 1, k_2, k_3, \cdots, k_r)$
  - using $k_1 = l, k_2 = 2, k_3 = k$ in the above
    $R(3, k, l) \geq 4R(k, l - 1) - 3$
  - use $k = 3$
    $R(3, 3, l) \geq 4R(3, l - 1) - 3$
- $R(3, 3, k) = \Theta(k^3 \text{poly-log}\, k)$
  Alon-Rődl 2005

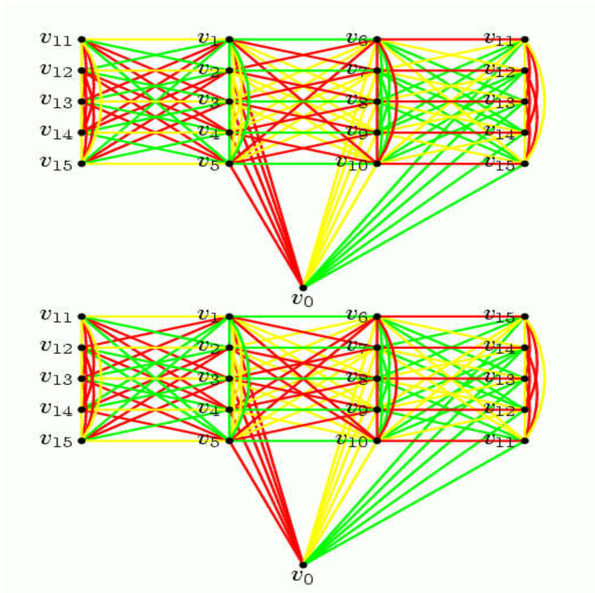
$R_r(3) = R(3, 3, \cdots, 3)$

just no triangles

- The limit $L = \lim_{r\to\infty} R_r(3)^{\frac{1}{r}}$ exists
  Chung-Grinstead 1983
  $(2s(r) + 1)^{\frac{1}{r}} = c_r \approx_{(r=6)} 3.199 < L$
- Much work on Schur numbers $s(r)$
  via sum-free partitions and cyclic colorings
  $s(r) > 89^{r/4 - c \log r} > 3.07^r$ [except small r]
  Abbott+ 1965+
- $s(r) + 2 \leq R_r(3)$
  $s(r) = 1, 4, 13, 44, \geq 160, \geq 536$
- $R_r(3) \geq 3R_{r-1}(3) + R_{r-3}(3) - 3$
  Chung 1973

$R(3,3,3) = 17$

two Kalbfleisch $(3,3,3;16)$-colorings, each color is a Clebsch graph



[Wikipedia]

Three colors - $R(3,3,4)$

the only (as of now) not hopeless case

- $30 \leq R(3,3,4)$, cyclic coloring, Kalbfleisch 1966
- $R(3,3,4) \leq 31$, computations, Piwakowski-R 1998

Theorem (Piwakowski-R 2001): $R(3,3,4) = 31$ iff there exists a $(3,3,4;30)$-coloring C in which every edge in 3-rd color has an endpoint x with degree 13. Furthermore, C has at least 25 vertices with color degree sequence $(8,8,13)$.

Proof: Gluing possible arrangements of color induced neighborhoods of v in a $(3,3,4;30)$-coloring:

$(3,4;s), (3,4;t), (3,3,3;u \geq 14)$ with $s+t+u = 29$

too many $(3,3,3;13)$'s to proceed further $\diamond$

Four colors - $R_4(3)$

$51 \le R(3,3,3,3) \le 62$

year	reference	lower	upper
1955	Greenwood, Gleason	42	66
1967	false rumors	[66]	
1971	Golomb, Baumert	46	
1973	Whitehead	50	65
1973	Chung, Porter	51	
1974	Folkman		65
1995	Sánchez-Flores		64
1995	Kramer (no computer)		62
2004	Fettes-Kramer-R (computer)		62

History of bounds on $R_4(3)$ [from FKR 2004]

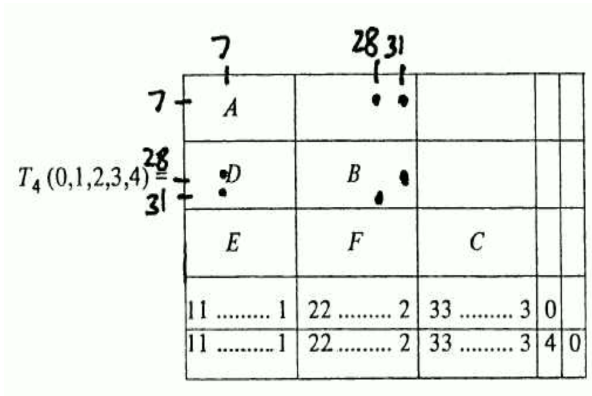
Lower bound for $R_4(3)$

start with Clebsch (3,3,3;16)-coloring

$$T_3(x_0,x_1,x_2,x_3) = \left|\begin{array}{l}
x_0 \\
x_1x_0 \\
x_1x_2x_0 \\
x_1x_2x_3x_0 \\
x_1x_3x_3x_2x_0 \\
x_1x_3x_2x_3x_2x_0 \\
x_2x_3x_2x_2x_1x_1x_0 \\
x_2x_2x_3x_1x_1x_2x_3x_0 \\
x_2x_2x_1x_3x_2x_1x_3x_1x_0 \\
x_2x_1x_1x_2x_3x_2x_1x_1x_3x_0 \\
x_2x_1x_2x_1x_2x_3x_1x_3x_1x_3x_0 \\
x_3x_2x_1x_1x_3x_3x_2x_3x_3x_2x_2x_0 \\
x_3x_1x_2x_3x_3x_1x_3x_1x_3x_2x_3x_1x_0 \\
x_3x_1x_3x_2x_1x_3x_3x_2x_1x_3x_2x_1x_2x_0 \\
x_3x_3x_3x_1x_2x_1x_2x_2x_3x_1x_3x_2x_2x_1x_0 \\
x_3x_3x_1x_3x_1x_2x_2x_3x_2x_3x_1x_2x_1x_2x_1x_0
\end{array}\right|$$

Lower bound for $R_4(3)$

Chung construction 1973, basic step yields (3,3,3,3;50)-coloring



$$A = T_3(0,2,3,4) \qquad D = T_3(3,2,1,4)$$
$$B = T_3(0,3,1,4) \qquad E = T_3(2,1,3,4)$$
$$C = T_3(0,1,2,4) \qquad F = T_3(1,3,2,4)$$

Lower bound for $R_4(3)$

attempts to beat Chung's construction for 4 colors

Iterate transformations of colorings:

- Merging pairs of colors (easy)

$$(3,3,3,3;n) \to (3,3,6;n), \quad (3,3,6;n) \to (6,6;n)$$

- Deleting a vertex with all adjacent edges (easy)

$$(3,3,3,3;n) \to (3,3,3,3;n-1)$$

- Single color splitting (moderate)

$$(6,6;n) \to (3,3,6;n), \quad (3,3,6;n) \to (3,3,3,3;n)$$

- Limited one point extension (hard)

$$(6,6;n) \to (6,6;n+1)$$

Lower bound for $R_4(3)$

attempts to beat Chung's construction

Results

- Many nonisomorphic constructions on 50 vertices, yet, all of them are just minor modifications of the Chung construction.
- Very hard to get close to 50 vertices with heuristics.

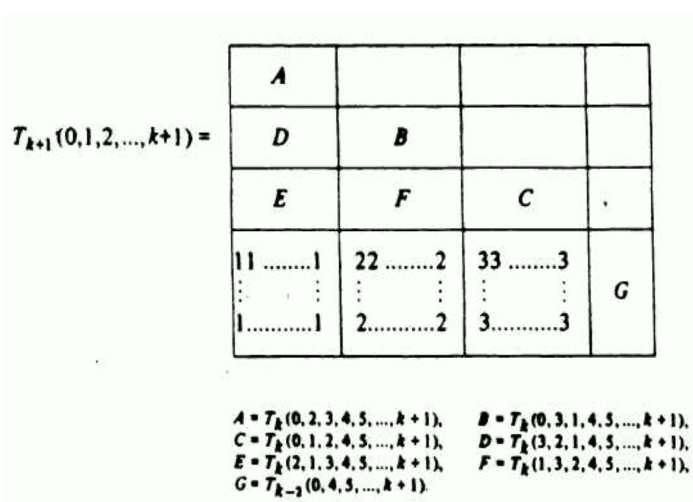
Used great software by Brendan McKay

- *nauty*, canonical labelings of graphs (and more), isomorph deletion
- *geng*, graph generator
- *autoson*, network job scheduling

Lower bound for $R_k(3)$

Chung construction 1973, recursion

$$R_r(3) \geq 3R_{r-1}(3) + R_{r-3}(3) - 3$$



Upper bound for $R_4(3)$

color degree sequences for $(3, 3, 3, 3; \geq 59)$-colorings

n	orders of $N_\eta(v)$	
65	[16, 16, 16, 16]	Whitehead, Folkman 1973-4
64	[16, 16, 16, 15]	Sánchez-Flores 1995
63	[16, 16, 16, 14]	
	[16, 16, 15, 15]	
62	[16, 16, 16, 13]	Kramer 1995+
	[16, 16, 15, 14]	–
	[16, 15, 15, 15]	Fettes-Kramer-R 2004
61	[16, 16, 16, 12]	
	[16, 16, 15, 13]	
	[16, 16, 14, 14]	
	[16, 15, 15, 14]	
	[15, 15, 15, 15]	
60	[16, 16, 16, 11]	guess: doable in 2015
	[16, 16, 15, 12]	
	[16, 16, 14, 13]	
	[16, 15, 15, 13]	
	[16, 15, 14, 14]	
	[15, 15, 15, 14]	
59	[16, 16, 16, 10]	
	[16, 16, 15, 11]	
	[16, 16, 14, 12]	
	[16, 16, 13, 13]	
	[16, 15, 15, 12]	
	[16, 15, 14, 13]	
	[15, 15, 15, 13]	
	[15, 15, 14, 14]	

More colors - summary

just no triangles

k	value	or	bounds	reference(s)
2		6		[cf. Bush 1953]
3		17		Greenwood-Gleason 1955
4	51	–	62	Chung 1973 – Fettes-Kramer-R 2004
5	162	–	307	Exoo 1994 – easy
6	538	–	1838	Fredricksen-Sweet 2000 – easy
7	1682	–	12861	Fredricksen-Sweet 2000 – easy

Bounds and values of $R_k(K_3)$

Things to do

computational multicolor Ramsey numbers problems

- improve $45 \leq R(3,3,5) \leq 57$
- finish off $30 \leq R(3,3,4) \leq 31$
- understand why heuristics don't find $51 \leq R_4(3)$
- improve on $R_4(3) \leq 62$

More Arrowing

F, G, H - graphs, s, t, s_i - positive integers

Definitions

$F \to (s_1, ..., s_r)^e$ iff for every r-coloring of the edges
F contains a monochromatic copy of K_{s_i} in some color i.

$F \to (G, H)^e$ iff for every blue/red edge-coloring of F,
F contains a blue copy of G or a red copy of H.

Facts

$R(s,t) = \min\{n \mid K_n \to (s,t)^e\}$
$R(G,H) = \min\{n \mid K_n \to (G,H)^e\}$

Folkman problems

edge Folkman graphs
$\mathcal{F}_e(s, t; k) = \{G \to (s, t)^e : K_k \not\subseteq G\}$

edge Folkman numbers (very hard to compute)
$F_e(s, t; k)$ = the smallest n such that there exists an n-vertex graph G in $\mathcal{F}_e(s, t; k)$

vertex Folkman graphs/numbers (hard to compute)
2-coloring vertices instead of edges

Theorem (Folkman 1970): For all $k > max(s, t)$, edge- and vertex Folkman numbers $F_e(s, t; k)$, $F_v(s, t; k)$ exist.

Two small cases

warming up

- $G = K_6$ has the smallest number of vertices among graphs which are not a union of two K_3-free graphs, or
    - $K_6 \to (K_3, K_3)^e$ and $K_5 \not\to (K_3, K_3)^e$
- What if we want G to be K_6-free? Graham (1968) proved that
    - $K_8 - C_5 = K_3 + C_5 \to (K_3, K_3)$
      $|V(H)| < 8 \wedge K_6 \not\subseteq H \Rightarrow H \not\to (K_3, K_3)$

Known values/bounds for $F_e(3,3;k)$

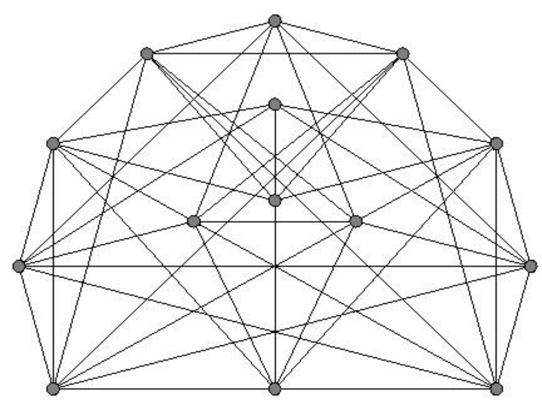
the challenge is to compute $F_e(3,3;4)$

$k > R(s,t) \Rightarrow F_e(s,t;k) = R(s,t)$
$k \le R(s,t)$, very little known in general

k	$F_e(3,3;k)$	graphs	reference
≥ 7	6	K_6	folklore
6	8	$C_5 + K_3$	Graham 1968
5	15	659 graphs	Piwakowski-Urbański-R 1999
4	≤ 941	$\alpha^5 \mod 941$	Dudek-Rődl 2008

$F_e(3,3;5) = 15$, and $F_v(3,3;4) = 14$

$G + x \to (3,3)^e$, and $G \to (3,3)^v$



unique 14-vertex bicritical $F_v(3,3;4)$-graph G [PRU 1999]

History of upper bounds on $F_e(3,3;4)$

- 1967 - Erdős, Hajnal state the problem
- 1970 - Folkman proves his theorem for 2 colors
  VERY large bound for $F_e(3,3;4)$.
- 1975 - Erdős offers \$100 (or 300 Swiss francs)
  for deciding if $F_e(3,3;4) < 10^{10}$
- 1988 - Spencer, probabilistic proof for the bound 3×10^8
  (1989 - Hovey finds a mistake, bound up to 3×10^9)
- 2007 - Lu, ≤ 9697, spectral analysis of modular circulants
- 2008 - Dudek-Rődl, $F_e(3,3;4) \leq 941$
  circulant arc lengths $\alpha^5 \mod 941$

$F_e(3,3;4) \leq 941$

some details of the proof by Dudek-Rődl

- **Theorem.** If for every vertex $v \in V(G)$

$$Maxcut(G[N(v)]) < \frac{2}{3}|E(G[N(v)])|$$

  then $G \rightarrow (3,3)^e$.

- Define graph H on vertices $E(G)$ with edges
  $\{(e,f) : e,f \in E(G), efg \text{ is a triangle in } G \text{ for some } g\}$.

  Maxcut approximation in H can imply $G \rightarrow (3,3)^e$.

- This works for the graph

$$G = (\mathcal{Z}_{941}, \{(i,j) : i - j = \alpha^5 \mod 941\})$$

History of lower bounds on $F_e(3,3;4)$

- $10 \le F_e(3,3;4)$ Lin 1972
- $16 \le F_e(3,3;4)$ Piwakowski-Urbański-R 1999
  since $F_e(3,3;5) = 15$, all graphs in $\mathcal{F}_e(3,3;5)$ on 15 vertices are known, and all of them contain K_4's
- $19 \le F_e(3,3;4)$ R-Xu 2007
  $18 \le F_e(3,3;4)$ proof "by hand"
- ANY proof technique improving on 19 very likely will be of interest

Lower bound

proof "by hand" that $18 \le F_e(3,3;4)$

- G_{17} critical for $R(4,4) = 18$, check that $G_{17} \not\to (3,3;4)^e$.
- $G_{17} \not\approx G \to (3,3;4)^e$, $|V(G)| = 17$, G must have indset I on 4 vertices.
- $H = I + G[V(G) \setminus I] \to (3,3;5)^e$.
- Dropping any three vertives from I, gives K_5-free graph on 14 vertices.
- Contradiction with $F_e(3,3;5) = 15$.

Computing $19 \le F_e(3,3;4)$
quite similar, but much more work,
use all 153 graph $H \in \mathcal{F}_v(3,3;4)$.

General facts on $F_e(s,t;k)$

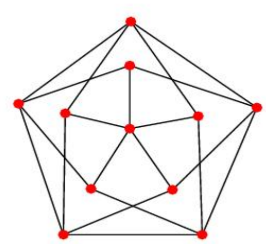
- $G \in \mathcal{F}_e(s,t;k) \Rightarrow \chi(G) \geq R(s,t)$
  no k in the bound!, easy

- $\mathcal{F}_e(s,t;k) = R(s,t)$ for $k > R(s,t)$ easy
- $\mathcal{F}_e(s,t;R(s,t)) = R(s,t) + c$ so, so
  in most cases c is small (2, 4, 5)
- $\mathcal{F}_e(s,t;k) \geq R(s,t) + 4$ for $k < R(s,t)$ hard

- $G \in \mathcal{F}_v(R(s-1,t), R(s,t-1); k-1) \Rightarrow$
  $G + x \in \mathcal{F}_e(s,t;k)$, or equivalently
- $G + x \not\rightarrow (s,t)^e \Rightarrow G \not\rightarrow (R(s-1,t), R(s,t-1))^v$,
  and clearly $cl(G+x) = cl(G) + 1$

Vertex Folkman numbers pearls

$F_v(2,2,2;3) = 11$
the smallest 4-chromatic triangle-free graph

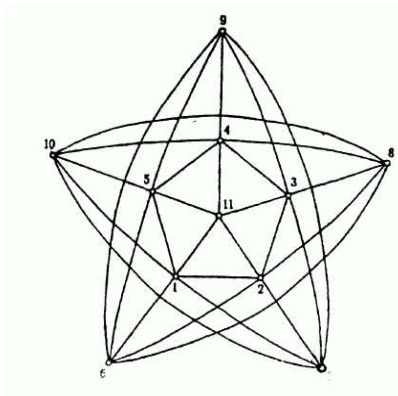


Grőtzsch graph [mathworld.wolfram.com]

$F_v(2,2,2,2;3) = 22$, Jensen-Royle 1995
the smallest 5-chromatic triangle-free graph has 22 vertices

Vertex Folkman numbers pearls

$F_v(2,2,2,2;4) = 11$, Nenov 1984, also 1993
the smallest 5-chromatic K_4-free graph has 11 vertices



$17 \leq F_v(4,4;5) \leq 23$, Xu-Luo-Shao 2010

Vertex Folkman numbers pearls

Theorem (ancient folklore)
$F_v(\underbrace{2,\cdots,2}_{r};r) = r+5$, for $r \geq 5$.

Proof. For the upper bound consider
as the critical graph $K_{r-5} + C_5 + C_5$
for the lower bound take any
K_r-free graph G on $r+4$ vertices, then
assemble matchings in $\overline{G}$ to show $\chi(G) \leq r$ □

Theorem (Nenov 2003)
$F_v(\underbrace{3,\cdots,3}_{r};2r) = 2r+7$, for $r \geq 3$.
For $r = 2$, a small but hard case, $F_v(3,3;4) = 14$ (PRU 1999)

Testing arrowing is hard

theory/practice

- Testing whether $F \to (3,3)^e$ is **coNP**-complete
  Burr 1976
- Determining if $R(G,H) < m$ is **NP**-hard
  Burr 1984
- Testing whether $F \to (G,H)^e$ is Π_2^p-complete
  Schaefer 2001
- Implementing fast $F \to (3,3)^e$ is challenging

Testing whether $F \to (K_2, K_n)^e$ is the same as checking $K_n \subset F$, so it is **NP**-hard.

Complexity of (edge) arrowing

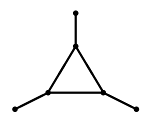
Problem	Fixed	Complexity
$F \to (G,H)$		Π_2^p-complete
$F \to (G,H)$	G, H	in **coNP**
$F \to (K_2,H)$		**NP**-complete
$F \to (K_2,H)$	H	**NP**-complete
$F \to (T,K_n)$	T, $e(T) \geq 2$	Π_2^p-complete
$F \to (G,H)$	$G, H \in \Gamma_3$	**coNP**-complete
$F \to (P_4,P_4)$		**coNP**-complete
$F \to (kK_2,H)$	k, H	P
$F \to (K_{1,n},K_{1,m})$		P
$K_n \to (G,H)$		**NP**-hard

Compendium of arrowing complexity by many.

Tools in complexity of arrowing

(G, H)-enforcers, -signal senders, -cleavers, -determiners are the tools (gadgets) used in reductions (Burr, Schaefer). They give control on $F \rightarrow (G, H)$.

Definition. (Grossman 1983) F is a (G, G)-cleaver iff there exists unique coloring of F witnessing $F \not\rightarrow (G, G)$.



P_4 cleaved $F \not\rightarrow (P_4, P_4)$, with only one witness coloring.

Known K_3-cleaved graphs contain K_4.
K_5 is not C_5-cleaved, P_3 cleaves C_{2n}.

G_{127}

Hill-Irving 1982

$G_{127} = (\mathcal{Z}_{127}, E)$
$E = \{(x, y) | x - y = \alpha^3 \pmod{127}\}$

Ramsey $(4, 12)$-graph, a color in $(4, 4, 4; 127)$
Exoo started to study if $G_{127} \rightarrow (3, 3)^e$

- 127 vertices, 2667 edges, 9779 triangles
- no K_4's, independence number 11, regular of degree 42
- vertex- and edge-transitive
- 5334 $(= 127 * 42)$ automorphisms
- $(127, 42, 11, \{14, 16\})$ - regularity
- K_{127} can be partitioned into three G_{127}'s

When to expect $G \rightarrow (3,3)^e$?

- G has a large number of triangles
- G has many small dense subgraphs
- Spencer's proof is far from useful for G_{127}

Conjecture. $G_{127} \rightarrow (3,3)^e$

Plan. Find a subgraph H, embedded in G in many places, so there is a small number of colorings witnessing $H \not\rightarrow (3,3)^e$.
Try to extend all (not many) colorings for $H \not\rightarrow (3,3)^e$ to G.

or, if this is too expensive ...

go via SAT ...

Reducing $\{G \mid G \not\rightarrow (3,3)^e\}$ to 3-SAT

edges in $G \longmapsto$ variables of ϕ_G
each (edge)-triangle xyz in $G \longmapsto$ add to ϕ_G

$$(x + y + z) \wedge (\overline{x} + \overline{y} + \overline{z})$$

Clearly,

$$G \not\rightarrow (3,3)^e \iff \phi_G \quad \text{is satisfiable}$$

For $G = G_{127}$, ϕ_G has 2667 variables and 19558 3-clauses, 2 for each of the 9779 triangles.

Note: By taking only the positive clauses, we obtain a reduction to ϕ'_G in NAE-3-SAT with half of the clauses.

Use SAT-solvers

SAT-solver competitions, 3 medals in 9 categories

$$(\text{random, crafted, industrial}) \times (\text{SAT, UNSAT, ALL})$$

SATzilla (UBC) - winner of 2007 and 2009 competitions
clasp (D), precosat (SF/A/NL) - winners of 2009 competition

The category we need: CRAFTED-UNSAT

Rsat, Picosat, Minisat, March_KS
other recent leading SAT-solvers

$G_{127} \to (3,3)^e$?

zChaff experiments on $\phi_{G_{127}}$

- Pick $H = G_{127}[S]$ on $m = |S|$ vertices.
  Use zChaff to split H:
  - $m \leq 80$, H easily splittable
  - $m \approx 83$, phase transition ?
  - $m \geq 86$, splitting H is very difficult
- $\#(\textit{clauses})/\#(\textit{variables}) = 7.483$ for G_{127}, far above conjectured phase transition ratio $r \approx 4.2$ for 3-SAT.
  It is known that

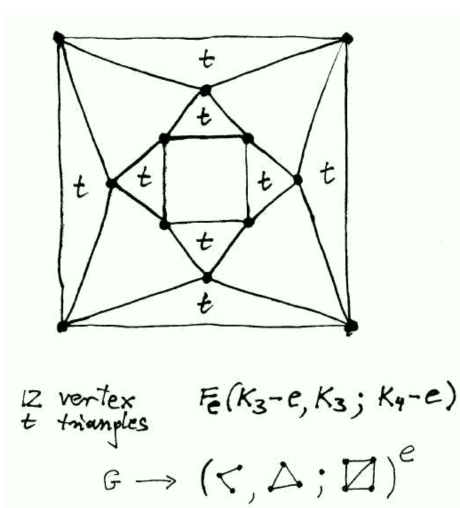
$$3.52 \leq r \leq 4.596$$

Folkman problems to work on

Is it true that $50 \le F_e(3,3;4) \le 100$?

- Decide whether $G_{127} \rightarrow (3,3)^e$
- Improve on $19 \le F_e(3,3;4) \le 941$
- Study $F_e(3,3;G)$ for $G \in \{K_5 - e, W_5 = C_4 + x\}$
- Study $F_e(K_4 - e, K_4 - e; K_4)$
- Don't study $F_e(K_3, K_3; K_4 - e)$
  it doesn't exist :–)

A small Folkman graph



can drop any vertex, arrowing still holds!

So, what to do next?

computationally

Hard but potentially feasible tasks:

- Improve any of the Ramsey bounds
  - $40 \le R(3, 10) \le 43$
  - $30 \le R(3, 3, 4) \le 31$
  - $51 \le R(3, 3, 3, 3) \le 62$
- Folkman arrowing of K_3
  - Improve on $19 \le F_e(3, 3; 4) \le 941$
  - Study $F_e(3, 3; G)$ for $G \in \{K_5 - e, W_5 = C_4 + x\}$

References

- Alexander Soifer
  The Mathematical Coloring Book, Springer 2009
- SPR's survey *Small Ramsey Numbers* at the *EIJC*
  Dynamic Survey DS1, revision #12, August 2009
  `http://www.combinatorics.org/Surveys`

All other references therein.

Thanks
for listening