Lower Bounds on Classical Ramsey Numbers

constructions, connectivity, Hamilton cycles

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Outline

Ramsey Numbers

- $R(G, H) = n$ iff
  minimal n such that in any 2-coloring of the edges of K_n there is a monochromatic G in the first color or a monochromatic H in the second color.
- $2 - colorings \cong graphs$, $R(m, n) = R(K_m, K_n)$
- Generalizes to k colors, $R(G_1, \cdots, G_k)$
- Theorem (Ramsey 1930): Ramsey numbers exist

Asymptotics
diagonal cases

- Bounds (Erdős 1947, Spencer 1975, Thomason 1988)

$$\frac{\sqrt{2}}{e} 2^{n/2} n < R(n,n) < \binom{2n-2}{n-1} n^{-1/2+c/\sqrt{\log n}}$$

- Newest upper bound (Conlon, 2010)

$$R(n+1,n+1) \leq \binom{2n}{n} n^{-c\frac{\log n}{\log\log n}}$$

- **Conjecture** (Erdős 1947, $100)
  $\lim_{n\to\infty} R(n,n)^{1/n}$ exists.
  If it exists, it is between $\sqrt{2}$ and 4 ($250 for value).

Asymptotics

Ramsey numbers avoiding K_3

- Recursive construction yielding
  $R(3, 4k+1) \geq 6R(3, k+1) - 5$
  $\Omega(k^{\log 6/\log 4}) = \Omega(k^{1.29})$
  Chung-Cleve-Dagum 1993

- Explicit $\Omega(k^{3/2})$ construction
  Alon 1994, Codenotti-Pudlák-Giovanni 2000

- Kim 1995, lower bound
  Ajtai-Komlós-Szemerédi 1980, upper bound
  Bohman 2009, triangle-free process

$$R(3, k) = \Theta\left(\frac{k^2}{\log k}\right)$$

Off-Diagonal Cases

fixing small k

- McKay-R 1995, $R(4,5) = 25$
- Bohman triangle-free process - 2009

$$R(4,n) = \Omega(n^{5/2}/\log^2 n)$$

- Kostochka, Pudlák, Rődl - 2010
  constructive lower bounds

$$R(4,n) = \Omega(n^{8/5}), \quad R(5,n) = \Omega(n^{5/3}), \quad R(6,n) = \Omega(n^2)$$

(vs. probabilistic $5/2, 6/2, 7/2$ with /logs)

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Values and Bounds on $R(k, l)$

two colors, avoiding cliques

k \ l	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 43	46 51	52 59	59 69	66 78	73 88
4		18	25	35 41	49 61	56 84	73 115	92 149	97 191	128 238	133 291	141 349	153 417
5			43 49	58 87	80 143	101 216	125 316	143 442	159	185 848	209	235 1461	265
6				102 165	113 298	127 495	169 780	179 1171	253	262 2566	317	5033	401
7					205 540	216 1031	233 1713	289 2826	405 4553	416 6954	511 10581	15263	22116
8						282 1870	317 3583	6090	10630	16944	817 27490	41525	861 63620
9							565 6588	580 12677	22325	39025	64871	89203	
10								798 23556		81200			1265

[EIJC survey *Small Ramsey Numbers, revision #12, 2009*]

General lower bound constructions
aren't that good

Theorem Burr, Erdős, Faudree, Schelp, 1989
$R(k,n) \geq R(k,n-1) + 2k - 3$ for $k \geq 2$, $n \geq 3$ (not $n \geq 2$)

Theorem (Xu-Xie-Shao-R 2004, 2010)
If $2 \leq p \leq q$ *and* $3 \leq k$*, then* $R(k, p+q-1) \geq$

$$R(k,p) + R(k,q) + \begin{cases} k-3, & \text{if} \quad 2 = p \\ k-2, & \text{if} \quad 3 \leq p \text{ or } 5 \leq k \\ p-2, & \text{if} \quad 2 = p \text{ or } 3 = k \\ p-1, & \text{if} \quad 3 \leq p \text{ and } 4 \leq k \end{cases}$$

For $p = 2$, $n = q + 1$, we have $R(k,p) = k$,
which implies BEFR'89

Proof by construction

Given

(k,p)-graph G, (k,q)-graph H, $k \geq 3$, $p,q \geq 2$
G and H contain induced K_{k-1}-free graph M

construct

$(k,p+q-1)$-graph F, $n(F) = n(G) + n(H) + n(M)$
$VG = \{v_1, v_2, ..., v_{n_1}\}$, $VH = \{u_1, u_2, ..., u_{n_2}\}$

$VM = \{w_1, ..., w_m\}$, $m \leq n_1, n_2$, $K_{k-1} \not\subset M$
$G[\{v_1, ..., v_m\}]$, $H[\{u_1, ..., u_m\}] \cong M$
$\phi(w_i) = v_i$, $\psi(w_i) = u_i$ isomorphisms

$VF = VG \cup VH \cup VM$
$E(G,H) = \{\{v_i, u_i\} \mid 1 \leq i \leq m\}$
$E(G,M) = \{\{v_i, w_j\} \mid 1 \leq i \leq n_1, 1 \leq j \leq m, \{v_i, v_j\} \in E(G)\}$
$E(H,M) = \{\{u_i, w_j\} \mid 1 \leq i \leq n_2, 1 \leq j \leq m, \{u_i, u_j\} \in E(H)\}$

.

Slow on citing this result...

In 1980, Paul Erdős wrote

Faudree, Schelp, Rousseau and I needed recently a lemma stating

$$\lim_{n\to\infty} \frac{r(n+1,n)-r(n,n)}{n} = \infty.$$

We could prove it without much difficulty, but could not prove that $r(n+1,n)-r(n,n)$ *increases faster than any polynomial of* n*. We of course expect*

$$\lim_{n\to\infty} \frac{r(n+1,n)}{r(n,n)} = C^{\frac{1}{2}},$$

where $C = \lim_{n\to\infty} r(n,n)^{1/n}$.

The best known lower bound for $(r(n+1,n)-r(n,n))$ is $\Omega(n)$.

Connectivity

Theorem 1

If $k \geq 5$ and $l \geq 3$, then the connectivity of any Ramsey-critical (k, l)-graph is no less than k.

This improves by 1 the result by Beveridge/Pikhurko from 2008

Hamiltonian cycles in Ramsey graphs

Theorem 2

If $k \geq l - 1 \geq 1$ and $k \geq 3$, except $(k, l) = (3, 2)$, then any Ramsey-critical (k, l)-graph is Hamiltonian.

In particular, for $k \geq 3$, all diagonal Ramsey-critical (k, k)-graphs are Hamiltonian.

Lower bound constructions

computer-free

Using the best known bounds for $R(k, s)$ we get:

Theorem 3

$$
\begin{aligned}
R(6,12) &\geq R(6,11) + 2 \times 6 - 2 &&\geq 263, \\
R(7,8) &\geq R(7,7) + 2 \times 7 - 2 &&\geq 217, \\
R(7,12) &\geq R(7,11) + 2 \times 7 - 2 &&\geq 417, \\
R(9,10) &\geq R(9,9) + 2 \times 9 - 2 &&\geq 581, \\
R(11,12) &\geq R(11,11) + 2 \times 11 - 2 &&\geq 1617, \\
R(12,12) &\geq R(12,11) + 2 \times 12 - 2 &&\geq 1639.
\end{aligned}
$$

Lower bound constructions

computer help

Theorem 4

$$\begin{aligned} R(5,17) &\geq 388, \\ R(5,19) &\geq 411, \\ R(5,20) &\geq 424, \\ R(6,8) &\geq 132, \\ R(7,9) &\geq 241, \\ R(8,17) &\geq 961, \\ R(8,8,8) &\geq 6079. \end{aligned}$$

What to do next?

Erdős and Sós, 1980, asked about

$3 \leq \Delta_k = R(3,k) - R(3,k-1) \leq k$:

$$\Delta_k \xrightarrow{k} \infty \, ? \qquad \Delta_k / k \xrightarrow{k} 0 \, ?$$

Challenges

- improve lower bound for Δ_k
- generalize beyond triangle-free graphs

Papers

SPR's papers to pick up

- Xu Xiaodong, Xie Zheng, SPR., A Constructive Approach for the Lower Bounds on the Ramsey Numbers $R(s, t)$, *Journal of Graph Theory*, 47 (2004), 231–239.
- Xiaodong Xu, Zehui Shao, SPR., More Constructive Lower Bounds ... (this talk), *SIAM Journal on Discrete Mathematics*, 25 (2011), 394–400.
- Revision #12 of the survey paper *Small Ramsey Numbers* at the *EIJC*, August 2009. Revision #13 coming in the summer 2011 ...