Bounds on Shannon Capacity and Ramsey Numbers

from Product of Graphs

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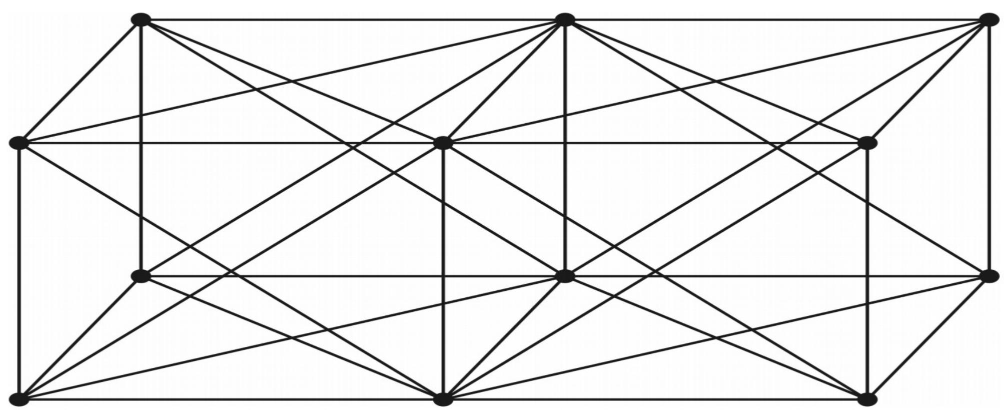
Outline

Strong Product of Graphs

- Graphs $G_i = (V_i, E_i)$, $1 \leq i \leq k$
- Strong graph product $G_1 \times \cdots \times G_k$

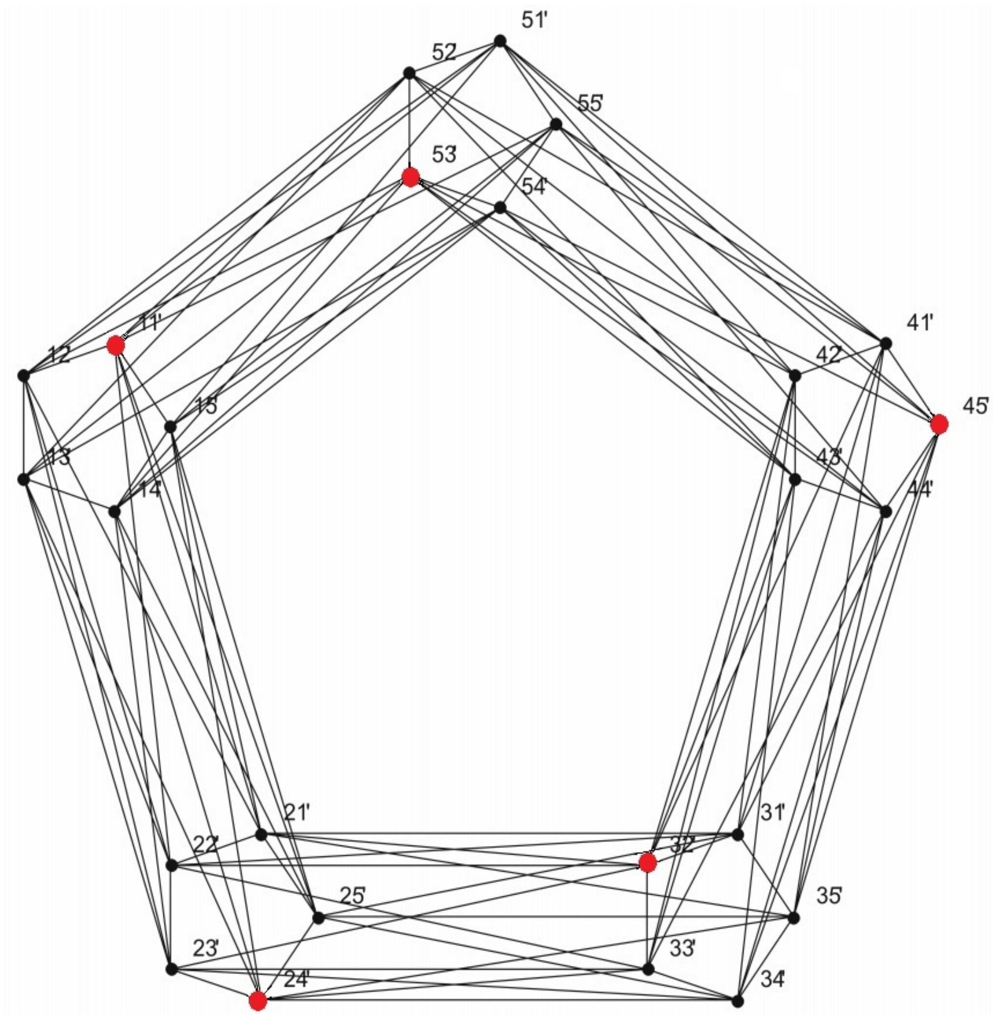
  Vertices: $V_1 \times \cdots \times V_k$

  Edges: $\{(u_1, \ldots, u_k), (v_1, \ldots, v_k)\}$,
  such that $u_i = v_i$ or $\{u_i, v_i\} \in E_i$, for each $1 \leq i \leq k$



$P_3 \times C_4$ [Fig. Jurkiewicz-Kubale 2011]

Strong Product of Graphs



$\alpha(C_5 \times C_5) = 5$ [Fig. Jurkiewicz-Kubale 2011]

Shannon Capacity
of a noisy channel modeled by graph

- vertices = transmitted characters
  edges = possible confusion
- Sending any independent set is safe.
  How well can we do with repeated use of the channel?
- Shannon (1956)
  The capacity of a noisy channel modeled by graph G:

$$c(G) = \lim_{n\to\infty} \alpha(G^n)^{1/n}$$

$\alpha(G^n)$ is the independence number of $G \times \cdots \times G$
vertices on each coordinate of any independent set in G^n induce an nonedge in G

Shannon Capacity of Cycles

better than naive for C_5

- $c(C_5) = \sqrt{5}$, (Lovász 1979)

  Consider cycle C_5, 1-2-3-4-5-1
  In C_5^2, {11, 32, 53, 24, 45} is an independent set

$$4 = \alpha(C_5)^2 < \alpha(C_5^2) = 5$$

- This inequality is "all we get" for C_5^t
- $c(C_7)$ is still unknown
- For even cycles $c(C_{2k}) = k$, since

$$k^t = \alpha(C_{2k})^t = \alpha(C_{2k}^t)$$

Shannon Capacity

difficult, interdisciplinary

- EE, communication, information, and coding theory
- Graph theory - papers by Alon, Bohman, Codenotti, Li, Lubetzky, Resta, and others on graph theoretic and combinatorial perspective

- In the strong product of graphs:

$$\alpha(G)\alpha(H) \leq \alpha(G \times H)$$

Lovász function

- Lovász $\vartheta(G)$ function, for $c \in R^N, |c| = 1, N \leq n(G)$:

$$\vartheta(G) = \min_{c,U} \max_{i \in V} \frac{1}{(c^T u_i)^2},$$

  where U is an orthonormal representation of G, i.e. $R^N \supset U = \{u_i\}_{i \in V}$, $u_i^T u_j = 0$ if $ij \notin E(G)$.
- $\vartheta(G)$ is relatively efficient (though not easy) to compute
- $\vartheta(G)\vartheta(H) = \vartheta(G \times H)$,
  $\vartheta(G)\vartheta(\overline{G}) \geq n(G)$
- Sandwich theorem:

$$\alpha(G) \leq c(G) \leq \vartheta(G) \leq \chi(\overline{G})$$

Shannon capacity of odd cycles

computational problem

- Vesel-Žerovnik 2000

  $108 \leq \alpha(C_7^4) \leq 115$
  $343 \leq \alpha(C_7^5) \leq 402$

  Improve on $343^{1/5} = 3.214... < c(C_7) < 3.318$

- Bohman-Holzman 2003
  good bounds for $\alpha(C_{2k+1}^d)$ for special d and k

  Improve on what is known on $c(C_{2k+1})$

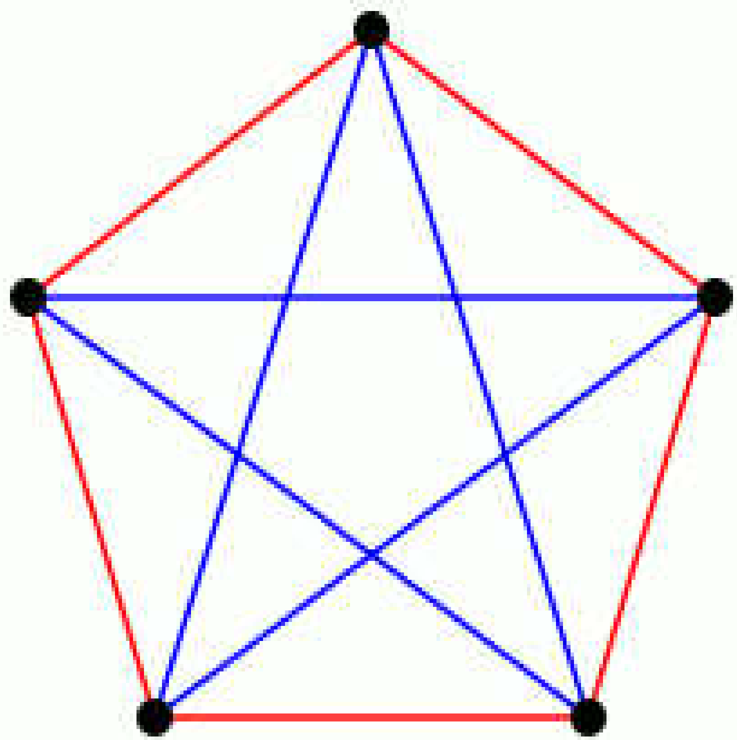
Ramsey Numbers

- $R(G, H) = n$ iff
  minimal n such that in any 2-coloring of the edges of K_n there is a monochromatic G in the first color or a monochromatic H in the second color

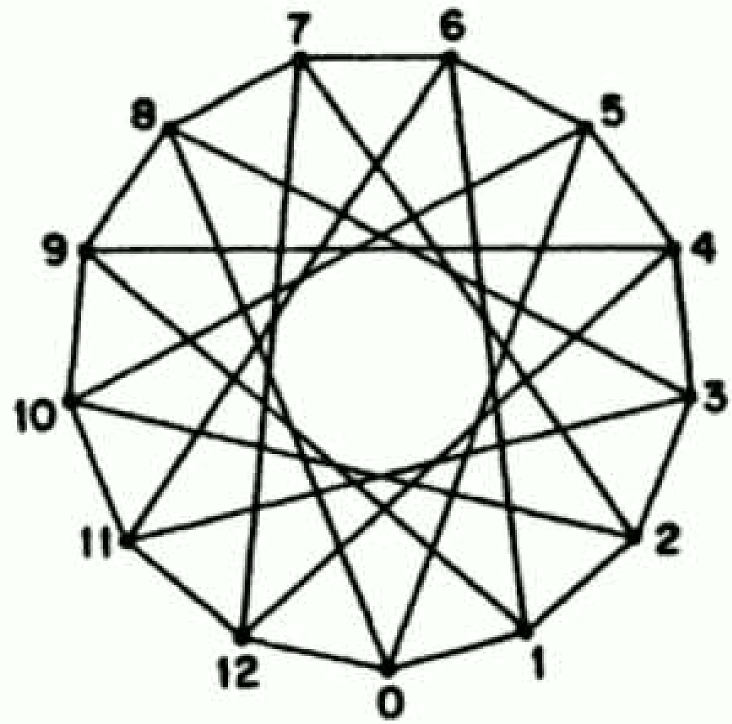
- *2-colorings* $\cong$ *graphs*, $R(m, n) = R(K_m, K_n)$
- Generalizes to k colors, $R(G_1, \cdots, G_k)$

- Theorem (Ramsey 1930): Ramsey numbers exist

Unavoidable classics



$R(3,3) = 6$

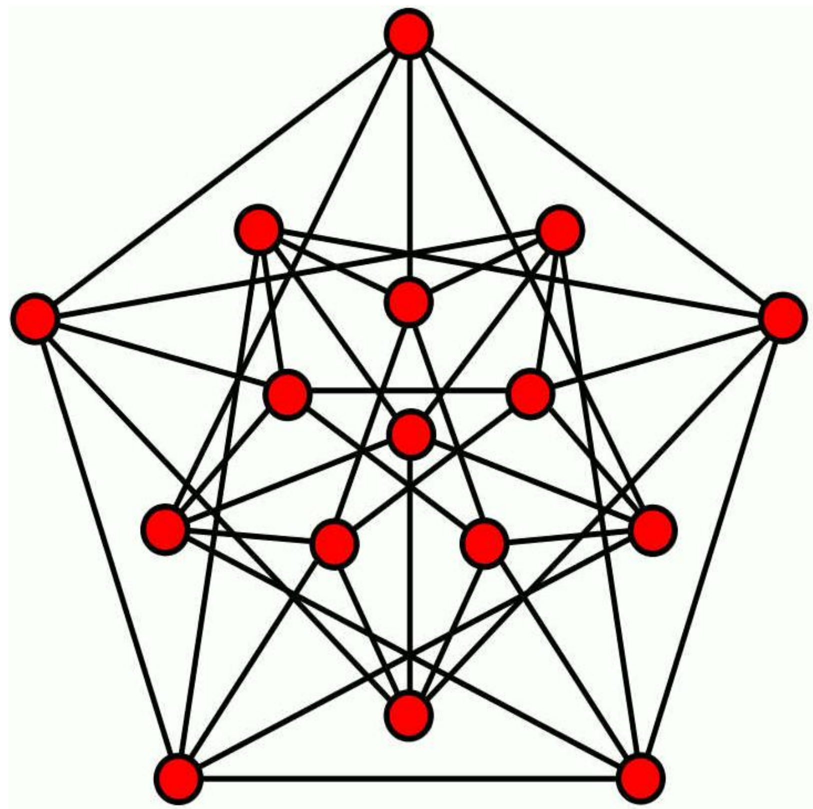


$R(3,5) = 14$ [GRS'90]

Clebsch $(3,6;16)$-graph on $GF(2^4)$

$(x,y) \in E$ iff $x - y = \alpha^3$

[Wikipedia]



Alfred Clebsch (1833-1872)

Asymptotics
diagonal cases

- Bounds - Erdős 1947, Spencer 1975, Conlon 2010

$$\frac{\sqrt{2}}{e} 2^{n/2} n < R(n,n) < R(n+1,n+1) < \binom{2n}{n} n^{-c\frac{\log n}{\log\log n}}$$

- **Conjecture** (Erdős 1947, $100)

$\lim_{n\to\infty} R(n,n)^{1/n}$ exists.

If it exists, it is between $\sqrt{2}$ and 4 ($250 for value).

- **Theorem** (Chung-Grinstead 1983)
$L = \lim_{k\to\infty} R_k(3)^{1/k}$ exists.

$3.199 < L$, (Fredricksen-Sweet 2000, X-Xie-Exoo-R 2004)

Things to do

computational multicolor Ramsey numbers problems

- Improve $45 \leq R(3,3,5) \leq 57$
- Finish off $30 \leq R(3,3,4) \leq 31$
- Improve on $R_4(3) \leq 62$,
  understand why heuristics don't find $51 \leq R_4(3)$
- Improve on $3.199 < \lim_{k\to\infty} R_k(3)^{1/k}$

Recall: $c(G) = \lim_{k\to\infty} \alpha(G^k)^{1/k}$

Adding many colors

to $R_n(k)$ constructions

- Abbott 1965, Song 1994

$$R_{n+m}(k) > \left(R_n(k) - 1\right)\left(R_m(k) - 1\right)$$

- $G[H]$ product of colorings

  Vertices: $V(G) \times V(H)$

  Edge colors:
  if $u_1 = u_2$ then $\{(u_1, v_1), (u_2, v_2)\}$ in $G[H]$ has color of $\{v_1, v_2\}$ in H, else color of $\{u_1, u_2\}$ in G

- $G[H]$ and $H[G]$ need not be isomorphic

Adding vertices

to union constructions

Construction (X-Xie-Exoo-R 2004, X-Shao-R 2011)
Given (k,p)-graph G, (k,q)-graph H, $k \geq 3$, $p,q \geq 2$, such that G and H contain a common induced K_{k-1}-free graph M, there exists $(k, p+q-1)$-graph F, $n(F) = n(G) + n(H) + n(M)$.

$VG = \{v_1, v_2, ..., v_{n_1}\}$, $VH = \{u_1, u_2, ..., u_{n_2}\}$
$VM = \{w_1, ..., w_m\}$, $m \leq n_1, n_2$, $K_{k-1} \not\subset M$
$G[\{v_1, ..., v_m\}]$, $H[\{u_1, ..., u_m\}] \cong M$
$\phi(w_i) = v_i$, $\psi(w_i) = u_i$ isomorphisms

$VF = VG \cup VH \cup VM$
$E(G,H) = \{\{v_i, u_i\} \mid 1 \leq i \leq m\}$
$E(G,M) = \{\{v_i, w_j\} \mid 1 \leq i \leq n_1, 1 \leq j \leq m, \{v_i, v_j\} \in E(G)\}$
$E(H,M) = \{\{u_i, w_j\} \mid 1 \leq i \leq n_2, 1 \leq j \leq m, \{u_i, u_j\} \in E(H)\}$

.

There exist multicolor extensions

Linking Shannon and Ramsey

Theorem

Erdős-McEliece-Taylor 1971

For arbitrary graphs $G_1, \ldots, G_n$,

$$\alpha(G_1 \times \cdots \times G_n) < R(\alpha(G_1) + 1, \ldots, \alpha(G_n) + 1)$$

and for all $k_1, \ldots, k_n > 0$ *there exist graphs* G_i *with* $\alpha(G_i) = k_i$, $1 \leq i \leq n$, *such that*

$$\alpha(G_1 \times \cdots \times G_n) = R(k_1 + 1, \ldots, k_n + 1) - 1.$$

For the diagonal case $k_i = k$, *there exists a graph* G *with* $\alpha(G) = k$, *such that* $\alpha(G^n) = R_n(k+1) - 1$.

Linking Shannon and Ramsey

- For all graphs G_i:
  $\alpha(G_1 \times \cdots \times G_n) < R(\alpha(G_1) + 1, \ldots, \alpha(G_n) + 1)$

  Make Ramsey witness on any MIS in $G_1 \times \cdots \times G_n$,
  edge $\{(u_1, \ldots, u_n), (v_1, \ldots, v_n)\}$ gets color i,
  the smallest index i with $u_i \neq v_i$.
- There exist graphs G_i:
  $R(k_1 + 1, \ldots, k_n + 1) = \alpha(G_1 \times \cdots \times G_n) + 1$

  Make G_i's from any Ramsey witness on $V = \{1, \ldots, m\}$,
  $G_i = (V, E_i)$, nonedges = color i, $\alpha(G_i) = k_i$,
  $\{(x, x, \ldots, x) : 1 \leq x \leq m\}$ is IS in $G_1 \times \cdots \times G_n$,
  $\alpha(G_1 \times \cdots \times G_n) = m$.

New Ramsey Lower Bound

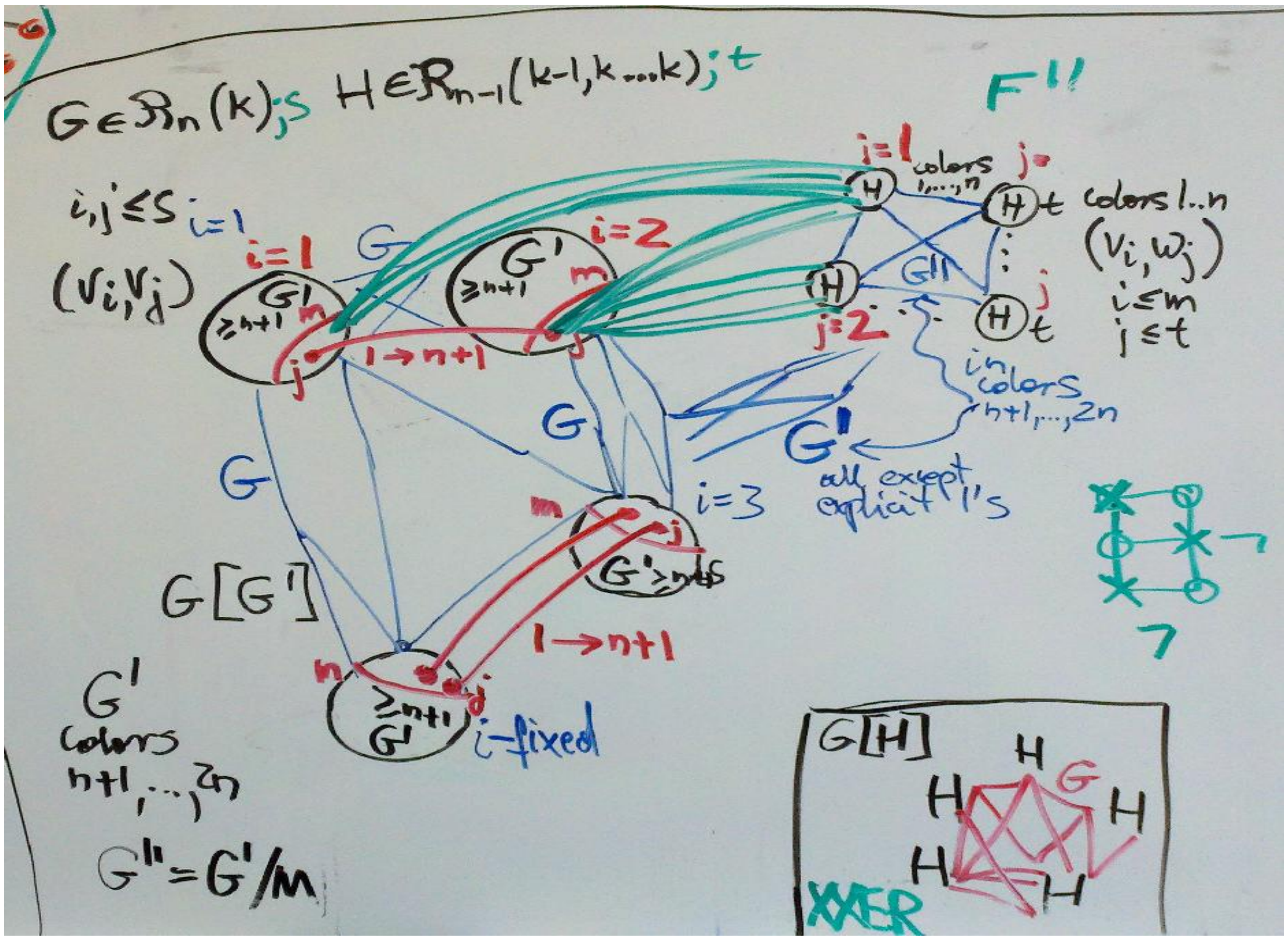
Theorem

For integers $k, n, m, s \geq 2$, let $G \in \mathcal{R}_n(k; s)$ be a coloring containing an induced subcoloring of K_m using less than n colors. Then

$$R_{2n}(k) \geq s^2 + m\big(R_n(k-1, \underbrace{k, \cdots, k}_{n-1}) - 1\big) + 1.$$

Proof. By construction

Proof by construction



Doubling colors, squaring+ vertices

Even better new Ramsey lower bound

Theorem

Let $G_0, G_1 \in \mathcal{R}_n(k; s)$, $H_0 \in \mathcal{R}_n(k-1, k, \ldots, k; t)$ be given for $n, k, s, t \geq 2$. Assume that G_1 contains an induced subcoloring of K_m using less than n colors, and that G_0 and H_0 both contain an induced subcoloring of K_r isomorphic to a coloring $H_1 \in \mathcal{R}_n(k-1, k, \ldots, k; r)$, for some $m \geq 2$, $r \geq 1$. Then

$$R_{2n}(k) \geq s^2 + m(t + r) + 1.$$

Problem.
Increase further RHS by a term improving the lower (constructive) asymptotics for $R_n(3)$ and $R_n(k)$.

Main Result

Theorem

If the supremum of the Shannon capacity $c(G)$ over all graphs G with $\alpha(G) = 2$ is finite and equal to C, then $C > \alpha(G^n)^{1/n}$ for any graph G with $\alpha(G) = 2$, for all $n > 0$.

Proof. Suppose that

$C^n = \alpha(G^n) < R_n(3)$, for some G with $\alpha(G) = 2$, and $n > 0$

There exists H with $\alpha(H) = 2$, $\alpha(H^{2n}) = R_{2n}(3) - 1$

By our constructions $\alpha(H^{2n}) > (R_n(3) - 1)^2 \geq \alpha(G^n)^2 = C^{2n}$

Hence, $c(H^{2n})^{1/2n} > C$, contradiction $\diamondsuit$

More general main result

Theorem

The supremum of the Shannon capacity over all graphs with bounded independence number cannot be achieved by any finite graph power.

Proof. Generalization of the last proof.

Main references

Xiaodong Xu, SPR, Bounds on Shannon Capacity and Ramsey Numbers from Product of Graphs, *IEEE Transactions on Information Theory*, 59(8) (2013) 4767–4770.

Our prior work and background:

- Xu Xiaodong, Xie Zheng, Geoffrey Exoo, SPR, Constructive Lower Bounds on Classical Multicolor Ramsey Numbers, *Electronic Journal of Combinatorics*, 11(1) (2004) #R35.
- Xiaodong Xu, Zehui Shao, SPR, More Constructive Lower Bounds on Classical Ramsey Numbers, *SIAM Journal on Discrete Mathematics*, 25 (2011), 394–400.
- SPR, *Small Ramsey Numbers*, *Electronic J. Combinatorics*, Dynamic Survey DS1, revision #14, January 2014.