# A Small Step Forwards on the Erdős-Sós Problem Concerning the Ramsey Numbers $R(3,k)$

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**Abstract**

Let $\Delta_s = R(K_3, K_s) - R(K_3, K_{s-1})$, where $R(G, H)$ is the Ramsey number of graphs $G$ and $H$ defined as the smallest $n$ such that any edge coloring of $K_n$ with two colors contains $G$ in the first color or $H$ in the second color. In 1980, Erdős and Sós posed some questions about the growth of $\Delta_s$. The best known concrete bounds on $\Delta_s$ are $3 \leq \Delta_s \leq s$, and they have not been improved since the stating of the problem. In this paper we present some constructions, which imply in particular that $R(K_3, K_s) \geq R(K_3, K_{s-1} - e) + 4$, and $R(3, K_{s+t-1}) \geq R(3, K_{s+1} - e) + R(3, K_{t+1} - e) - 5$ for $s, t \geq 3$. This does not improve the lower bound of 3 on $\Delta_s$, but we still consider it a step towards to understanding its growth. We discuss some related questions and state two conjectures involving $\Delta_s$, including the following: for some constant $d$ and all $s$ it holds that $\Delta_s - \Delta_{s+1} \leq d$. We also prove that if the latter is true, then $\lim_{s\to\infty} \Delta_s / s = 0$.

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# 1 Notation and Overview

In this paper all graphs are simple and undirected. The vertex set of graph $G$ is denoted by $V(G)$, $n(G) = |V(G)|$, the edge set by $E(G)$, and the set of neighbors of a vertex $v$ in $G$ will be written as $N_G(v)$. The independence number of $G$, denoted by $\alpha(G)$, is the order of the largest independent set in $G$. The graph induced in $G$ by the set of vertices $S \subset V(G)$ will be denoted by $G[S]$. For $v \in V(G)$ and $e \in E(G)$, $G - v$ will stand for $G[V \setminus \{v\}]$, and $G - e$ for the graph $G$ with edge $e$ removed.

For graphs $G$ and $H$, the *Ramsey number* $R(G, H)$ is the smallest positive integer $n$ such that every coloring of the edges of $K_n$ with two colors contains a monochromatic $G$ in the first color or a monochromatic $H$ in the second color. If the edges in the first color are interpreted as a graph $F$ and those in the second color as its complement, then $R(G, H)$ can be defined equivalently as the smallest $n$ such that every $G$-free graph on $n$ vertices contains $H$ in the complement. If $G = K_s$ and $H = K_t$ then we will write $R(s, t)$ for $R(G, H)$. Any $G$-free graph $F$ on $n$ vertices without $H$ in the complement will be called a $(G, H; n)$-graph. An $(s, t; n)$-graph will mean the same as a $(K_s, K_t; n)$-graph. A regularly updated survey by the third author [16] lists the values and the best known bounds on various types of Ramsey numbers.

In the sequel we will be concerned almost exclusively with the Ramsey numbers $R(3, G)$ and $(3, G; m)$-graphs for $G$ being $K_s$ or $K_s - e$. Observe that $R(3, G) = m + 1$ if and only if $m$ is the largest integer such that there exists a $(3, G; m)$-graph. Note also that in triangle-free graphs the neighborhoods are independent sets.

The asymptotics of $R(3, s)$ was extensively studied and now it is quite well understood. It is known that

$$\Big(1/4 + o(1)\Big)\frac{s^2}{\log s} \leq R(3, s) \leq \Big(1 + o(1)\Big)\frac{s^2}{\log s}.$$

In 1995, Kim [12] using probabilistic method improved lower bound asymptotics to $R(3, s) = \Omega(s^2/\log s)$. More detailed work followed, and finally the lower bound constant 1/4 was obtained recently by Bohman and Keevash [2], and independently by Fiz Pontiveros, Griffiths and Morris [8]. The upper bound constant 1 is implicit in a 1983 paper by Shearer [17], and it also can be stated without $o(1)$ for $s \geq 3$ as

$$R(3, s + 1) \leq \frac{(s - 1)^2}{\log s - 1 + s^{-1}} + 1. \tag{1}$$

However, the difference $R(3, G) - R(3, H)$ for concrete "consecutive" $G$ and $H$ is still very difficult to estimate, even starting with rather small cases. In general, for $K_s$ and $K_s - e$, all we know is the following:

Easy old bounds [3], see Section 2 and Construction 1 in Section 3,

$$3 \leq R(3, K_s) - R(3, K_{s-1}) \leq s, \tag{2}$$

trivial bounds implied by the monotonicity of Ramsey numbers

$$R(3, K_{s-1}) \leq R(3, K_s - e) \leq R(3, K_s), \tag{3}$$

and a result obtained in this paper (Corollary 7 in Section 4)

$$4 \leq R(3, K_{s+1}) - R(3, K_s - e). \tag{4}$$

Many attempts were made to improve on some part of (2) or (3), to no avail. We believe that our relatively simple constructions proving inequality (4) in Section 4 form an interesting step towards a better understanding of both (2) and (3). We pose it as a challenge to improve over any of the inequalities in (2), (3) or (4), or their combination as (4) combines parts of (2) and (3).

## 2 Erdős-Sós Problem

**Problem.** Erdős-Sós 1980 [7, 5]
*Let $\Delta_s = R(3, s) - R(3, s-1)$. Is it true that*

$$\Delta_s \stackrel{s\to\infty}{\longrightarrow} \infty \; ? \qquad (\Delta_s/s) \stackrel{s\to\infty}{\longrightarrow} 0 \; ? \tag{5}$$

Only easy bounds on $\Delta_s$ as in (2) are known. The upper bound $\Delta_s \leq s$ is obvious since the maximum degree of $(3, s)$-graphs is at most $s-1$. The lower bound $3 \leq \Delta_s$ looks misleadingly simple, but it is not trivial (see Construction 1 in the next section). It was argued in [10] that a better understanding of $\Delta_s$ may come from the study of $R(3, K_s - e)$ relative to $R(3, K_s) = R(3, s)$, since

$$\Delta_s = \Big(R(3, K_s) - R(3, K_s - e)\Big) + \Big(R(3, K_s - e) - R(3, K_{s-1})\Big).$$

Recent progress on what we know for small cases is significant [9, 10], however some very simple-looking questions remain open. For example, we do not even know whether $R(3, K_s - e) - R(3, K_{s-1})$ is positive for every large $s$. However, in Section 4 we prove (4), and in Section 5 we show that the second part of (5) holds under the assumption that there exists a constant $d$ for which $\Delta_s - \Delta_{s+1} \leq d$ for all $s$.

# 3 Previous Constructions

Burr, Erdős, Faudree and Schelp [3] in 1989 gave a general lower bound construction yielding $R(k, s+1) \geq R(k, s) + 2k - 3$ for $k, s \geq 2$. For $k = 3$ it is equivalent to the following construction, which implies $\Delta_s \geq 3$.

**Construction 1.** [3]
*For $s \geq 2$, given any $(3, s; n)$-graph, we can extend it to a $(3, s+1; n+3)$-graph.*

**Proof.** Let $u \in V(G)$ be any vertex of a $(3, s; n)$-graph $G$. Note that since $\alpha(G) < s$ we have $\deg_G(u) < s$. A $(3, s+1; n+3)$-graph $G'$ extending $G$ is defined on the set of 3 more vertices $V(G') = V(G) \cup \{v, x, y\}$ with the set of edges $E(G') = E(G) \cup \{vw \mid uw \in E(G)\} \cup \{ux, xy, yv\}$. Consider any independent set $I$ in $G'$, and the cardinality $t$ of its intersection with $\{u, v, x, y\}$. If $t \leq 1$ then $|I| \leq \alpha(G) + 1$, otherwise $t = 2$ and we must have that at least one of the vertices $u$ and $v$ is in $I$. Thus $|I \setminus \{u, v, x, y\}| < s - 1$, and hence $G'$ is a $(3, s+1; n+3)$-graph. $\diamond$

Theorem 2 below is based on a generalization of Construction 1, which together with a similar construction in [20] imply the lower bounds in the following Theorem 3 [19].

**Theorem 2.** [20] *For every $k \geq 3$ and $s, t \geq 2$, given any $(k, s)$-graph $G$ and $(k, t)$-graph $H$, if both $G$ and $H$ contain an induced subgraph isomorphic to some $K_{k-1}$-free graph $M$, then*

$$R(k, s+t-1) \geq n(G) + n(H) + n(M) + 1.$$

**Theorem 3.** [19]
*If $2 \leq s \leq t$ and $k \geq 3$, then*

$$R(k, s+t-1) \geq R(k, s) + R(k, t) + \begin{cases} k-3, & \text{if } s = 2; \\ k-2, & \text{if } s \geq 3 \text{ or } k \geq 5. \end{cases}$$

Gyárfás, Sebő and Trotignon [11] studied the growth of $R(3, s)$ in order to give precise bounds on the so called chromatic gaps. In particular, using Theorems 2 and 3, they describe various implications characterizing $R(3, s+k) - R(3, s)$.

# 4 New Constructions

We present two simple constructions, the second one generalizing the first, which together apparently add some new understanding of (2) and (3) and how they imply (4).

**Construction 4.** *For $s, t \geq 3$, given any $(3, s+1; m)$-graph $G$ and $(3, t+1; n)$-graph $H$, we construct from $G$ and $H$ a $(3, s+t; m+n-2)$-graph $F$.*

**Proof.** Let $G$ be any $(3, s+1; m)$-graph and $H$ any $(3, t+1; n)$-graph, on disjoint sets of vertices, and consider arbitrary two vertices $u \in V(G)$ and $v \in V(H)$. We will construct a $(3, s+t; m+n-2)$-graph $F$ on the vertex set $V(F) = V(G) \cup V(H) \setminus \{u, v\}$. Denote $X_G = N_G(u)$, $Y_G = V(G-u) \setminus X_G$, $X_H = N_H(v)$ and $Y_H = V(H-v) \setminus X_H$, so $V(F)$ is partitioned into $X_G \cup Y_G \cup X_H \cup Y_H$. The set of edges of $F$ is defined by $E(F) = E(G-u) \cup E(H-v) \cup \{xy \mid x \in X_G, y \in X_H\}$. Clearly, graph $F$ is triangle-free and it has the right number of vertices. We need to show that $\alpha(F) < s+t$. Note that $F$ contains a complete bipartite graph with partite sets $X_G$ and $X_H$, and thus for any independent set $I$ in $F$ we have $x_G = |I \cap X_G| = 0$ or $x_H = |I \cap X_H| = 0$, and also it holds that $y_G = |I \cap Y_G| \leq s-1$ and $y_H = |I \cap Y_H| \leq t-1$. Since $|I \cap V(G)| \leq s$ and $|I \cap V(H)| \leq t$, for $x_G > 0$ we have $|I| \leq s + y_H$, and for $x_H > 0$ we have $|I| \leq y_G + t$. In both cases $|I| \leq s+t-1$ as required. $\diamond$

We observe that Construction 4 (with adjusted $s$ and $t$) gives an alternate proof of a part of Theorem 3 for $k=3$, but also $R(3, s+t) \geq R(3, s+1) + R(3, t+1) - 3$. The latter is equivalent to Lemma 3.20 in [11], but we note that our path to this inequality is much simpler, and it does not depend on any intermediate results. In the next construction we exploit in more detail the structure of the base graphs $G$ and $H$.

**Construction 5.** *For $s, t \geq 3$, given any $(3, s+1; m)$-graph $G$ which has two nonadjacent vertices with at most $c_G$ common neighbors, and any $(3, t+1; n)$-graph $H$ which has two nonadjacent vertices with at most $c_H$ common neighbors, we construct from $G$ and $H$ a $(3, s+t-1; m+n-c_G-c_H-4)$-graph $F$.*

**Proof.** Let $G$ and $H$ be as stated above, where $u_1, u_2$ have $c_G$ common neighbors in $G$ and $v_1, v_2$ have $c_H$ common neighbors in $H$, respectively. We partition the set $V(G) \setminus \{u_1, u_2\}$ into $X_G^{12} \cup X_G^1 \cup X_G^2 \cup Y_G$, where $X_G^{12} = N_G(u_1) \cap N_G(u_2)$, $X_G^1 = N_G(u_1) \setminus N_G(u_2)$, $X_G^2 = N_G(u_2) \setminus N_G(u_1)$, and $Y_G = \{u \in V(G) \setminus \{u_1, u_2\} \mid uu_1 \notin E(G) \text{ and } uu_2 \notin E(G)\}$. Similarly, we set the partition $V(H) \setminus \{v_1, v_2\} = X_H^{12} \cup X_H^1 \cup X_H^2 \cup Y_H$ by considering four possible adjacencies to vertices $v_1, v_2$ in $H$. Obviously, $|X_G^{12}| = c_G$ and $|X_H^{12}| = c_H$. We will construct graph $F$ on the set of vertices $X_G^1 \cup X_G^2 \cup Y_G \cup X_H^1 \cup X_H^2 \cup Y_H$, which has cardinality as needed. The set of edges of $F$ is defined by

$$E(F) = E(G[X_G^1 \cup X_G^2 \cup Y_G]) \cup E(H[X_H^1 \cup X_H^2 \cup Y_H]) \cup K(X_G^1, X_H^1) \cup K(X_G^2, X_H^2),$$

where $K(X_G^1, X_H^1)$ and $K(X_G^2, X_H^2)$ are the edges of two complete bipartite graphs between indicated pairs of sets. It remains to be shown that $\alpha(F) \leq s+t-2$. Let $I \subset V(F)$ be any independent set in $F$, and denote by $x_G^1, x_G^2, y_G, x_H^1, x_H^2, y_H$ the orders of intersection of $I$ with the corresponding parts of $V(F)$. Similarly as in Construction 4, we have ($x_G^1 = 0$ or $x_H^1 = 0$) and ($x_G^2 = 0$ or $x_H^2 = 0$). Furthermore, $x_G^1 + y_G, x_G^2 + y_G \leq s-1$, $y_G \leq s-2$, and $x_H^1 + y_H, x_H^2 + y_H \leq t-1$, $y_H \leq t-2$. If $x_G^1 > 0$ and $x_G^2 > 0$, then $x_H^1 = x_H^2 = 0$ and thus $|I| \leq \alpha(G) + y_H \leq s + (t-2)$. If $x_H^1 > 0$ and $x_H^2 > 0$, then $x_G^1 = x_G^2 = 0$ and thus $|I| \leq \alpha(H) + y_G \leq t + (s-2)$. If only one of $x_G^1, x_G^2$ is positive, say $x_G^1 > 0$, then $x_G^2 = x_H^1 = 0$ and $|I| \leq (x_G^1 + y_G) + (x_H^2 + y_H) \leq (s-1) + (t-1)$. $\diamond$

One can look at Construction 5 as lowering the independence number of a union of $G$ and $H$ by 2, but at the cost of dropping $c_G + c_H + 4$ vertices. In the next three corollaries

| $s$ | $R(3,J_s)$ | $R(3,K_s)$ | $\Delta_s$ | $s$ | $R(3,J_s)$ | $R(3,K_s)$ | $\Delta_s$ |
|---|---|---|---|---|---|---|---|
| 3 | 5 | 6 | 3 | 10 | 37 | 40–42 | 4–6 |
| 4 | 7 | 9 | 3 | 11 | 42–45 | 47–50 | 5–10 |
| 5 | 11 | 14 | 5 | 12 | 47–53 | 53–59 | 3–12 |
| 6 | 17 | 18 | 4 | 13 | 55–62 | 60–68 | 3–13 |
| 7 | 21 | 23 | 5 | 14 | 60–71 | 67–77 | 3–14 |
| 8 | 25 | 28 | 5 | 15 | 69–80 | 74–87 | 3–15 |
| 9 | 31 | 36 | 8 | 16 | 74–91 | 82–98 | 3–16 |

Table 1: $R(3,J_s)$ and $R(3,K_s)$, $J_s = K_s - e$, for $s \leq 16$ [10], and [13, 14].

we show how in some cases we can further assume that $c_G = c_H = 0$. We will say that a $(3,s)$-graph $G$ is *edge minimal* if deletion of any of its edges increases $\alpha(G)$ to $s$, and it is *edge maximal* if addition of any edge creates a triangle. A $(3,s)$-graph $G$ is called *bicritical* if it is both edge minimal and maximal.

**Corollary 6.** For $s \geq 3$ and $m = R(3,s+1) - 1$, if there exists a $(3,s+1;m)$-graph which is not bicritical, then $\Delta_{s+2} \geq 4$.

**Proof.** Let $G'$ be a $(3,s+1;m)$-graph which is not bicritical. If it is not edge minimal, then the removal of some edge $e = uv \in E(G)$ gives a $(3,s+1;m)$-graph $G = G' - e$, in which vertices $u$ and $v$ have no common neighbors. If $G'$ is not edge maximal, let $G = G'$. In either case we have a $(3,s+1;m)$-graph $G$ with $c_G = 0$ which will be used with Construction 5.

There exist three $(3,4;8)$-graphs (with 10, 11 and 12 edges). Let $H$ be the well known unique $(3,4;8)$-graph with 10 edges, which is the cycle on 8 vertices $v_1v_2\cdots v_8$ with two consecutive main diagonal edges $v_1v_5$ and $v_2v_6$. The vertices $v_3$ and $v_7$ have no common neighbors. We will use this $H$ with $n = 8,\ t = 3$ and $c_H = 0$. The graph $F$ resulting from $G$ and $H$ by applying Construction 5 is a $(3,s+2;m+4)$-graph, which proves the claim that $\Delta_{s+2} \geq 4$. $\diamond$

Table 1 presents known values and bounds on $R(3,K_s)$, $R(3,K_s-e)$ collected in [9, 10] and $\Delta_s$ for $s \leq 16$, with an addition of recent improvements to the lower bounds on $R(3,K_s)$ for $12 \leq s \leq 15$ [13, 14] (which implies new lower bounds for $R(3,K_s-e)$ for $s = 14, 16$). We note that for $s \leq 9$, i.e. for which the exact value of $R(3,s)$ is known, there exist non-bicritical $(3,s;R(3,s)-1)$-graphs for $s \in \{4,6,7,8\}$.

**Corollary 7.** $R(3,s+1) \geq R(3,K_s-e) + 4$, for $s \geq 2$.

**Proof.** Let $m = R(3,K_s-e)-1$. Observe that every $(3,K_s-e;m)$-graph is a $(3,K_s;m)$-graph after removal of any of its edges, furthermore the endpoints of the removed edge share no common neighbors, since otherwise the original graph would have a triangle. Using Construction 5 for any such $(3,K_s)$-graph as $G$, and the $(3,4;8)$-graph $H$ as in the proof of Corollary 6, gives a $(3,s+1;m+4)$-graph $F$, which proves the lower bound. $\diamond$

In Table 1, for cases when only the bounds are given for $R(3, K_s - e)$ ($J_s = K_s - e$) or $R(3, K_s)$, we believe that the exact values are much closer to lower bounds than upper bounds. Actually, we expect that in most open cases the exact values are equal to the listed lower bounds. The exceptions, if any, likely include some of the lower bounds for $R(3, K_s - e)$, $s \in \{12, 14, 16\}$, which currently are the only cases in the scope of Table 1 when they are the same as the best known lower bounds for $R(3, K_{s-1})$.

We end this section with one more corollary, which is a little more general than Corollaries 6 and 7. Theorem 3 for $s \geq k = 3$ gives the inequality $R(3, s + t - 1) \geq R(3, s) + R(3, t) + 1$. The following Corollary 8 increases two terms of its right hand side and decreases the constant only by 6.

**Corollary 8.** $R(3, s + t - 1) \geq R(3, K_{s+1} - e) + R(3, K_{t+1} - e) - 5$, for $s, t \geq 3$.

**Proof.** Let $m = R(3, K_{s+1} - e) - 1$ and $n = R(3, K_{t+1} - e) - 1$. Consider any $(3, K_{s+1} - e; m)$-graph $G'$ and any $(3, K_{t+1} - e; n)$-graph $H'$. As in the proof of Corollary 7, let $G = G' - e$ for some edge $e \in E(G')$, then $G$ is a $(3, K_{s+1}; m)$-graph which has two nonadjacent vertices without common neighbors. Similarly, obtain $(3, K_{t+1}; m)$-graph $H$ from $H'$. Now, by applying Construction 5 to graphs $G$ and $H$ we obtain graph $F$ witnessing the claimed lower bound. $\diamond$

# 5 Two Conjectures

Observe that

$$R(3, s + k) - R(3, s - 1) = \sum_{i=0}^{k} \Delta_{s+i}. \tag{6}$$

We expect $\Delta_s$ to grow similarly as $s/\log s$ to account for the asymptotics of $R(3, s)$ known to be $\Theta(s^2/\log s)$, though with some small perturbations. $\Delta_s$ is actually known to be nonmonotonic as can be seen in Table 1 for $s$ between 4 and 6. However, we believe that such oscillations are contained as stated in the following Conjecture 9, where we anticipate that the decrease between consecutive $\Delta_s$ is bounded by a constant.

Gyárfás, Sebő and Trotignon [11] in their study of chromatic gaps, using Theorems 2 and 3, showed that we can obtain lower bounds on $R(3, s + k) - R(3, s)$ better than the obvious $3k$, for $k \geq 2, s \geq 3$. In particular, we have $\Delta_s \geq 3$, $\Delta_s + \Delta_{s+1} \geq 7$ and $\Delta_s + \Delta_{s+1} + \Delta_{s+2} \geq 11$.

**Conjecture 9.** *There exists $d \geq 2$ such that for all $s \geq 2$ we have $\Delta_s - \Delta_{s+1} \leq d$.*

Clearly, if $\Delta_s$ is nondecreasing for large $s$ then $\lim_{s\to\infty} \Delta_s = \infty$, but even if we could prove Conjecture 9 with $d = 1$ for $s$ sufficiently large (note that $\Delta_9 - \Delta_{10} \geq 2$), it is not clear that it would help to prove $\lim_{s\to\infty} \Delta_s = \infty$. However, we will show that if Conjecture 9 is true then it implies a positive solution to the second part of the Erdős-Sós problem.

**Theorem 10.** *If Conjecture 9 is true, then* $\lim_{s\to\infty} \Delta_s/s = 0$.

**Proof.** For contradiction, suppose that Conjecture 9 holds, but there exists $\epsilon > 0$ such that $\Delta_s \geq \epsilon s$ for infinitely many $s$, furthermore satisfying $s \geq 2d/\epsilon$. Note that the latter implies $\epsilon s/d - 2 \geq 0$. Define $k = \lfloor \epsilon s/d \rfloor$, then observe that $k \geq 2$, $k+1 > \epsilon s/d$ and $\epsilon s - kd \geq 0$. Now, assuming Conjecture 9, we have $\Delta_{s+i} \geq \epsilon s - id$ for $0 \leq i \leq k$, and using (6) we obtain the bound

$$R(3, s+k) - R(3, s-1) \geq (k+1)(\epsilon s - kd/2),$$

which gives

$$R(3, s+k) > \epsilon^2 s^2/2d. \tag{7}$$

On the other hand, the bound (1) with the above constraints on $s, \epsilon$ and $d$ implies

$$\begin{aligned} R(3, s+k) &\leq \frac{(s+k-2)^2}{\log(s+k-1) - 1 + (s+k-1)^{-1}} + 1 \\ &< \frac{(s+\epsilon s/d-2)^2}{\log(s+\epsilon s/d-2) - 1} + 1 \\ &< s^2 \left( \frac{(1+\epsilon/d)^2}{\log s - 1} + \frac{1}{s^2} \right) = s^2 f(s, \epsilon, d), \end{aligned}$$

where for fixed $\epsilon$ and $d$ we have $\lim_{s\to\infty} f(s, \epsilon, d) = 0$. This contradicts inequality (7) for $s$ large enough, and hence $\lim_{s\to\infty} \Delta_s/s = 0$. $\diamond$

Our attempts to improve on Theorem 10 led to the following stronger statement. Suppose that $a, b \in (0, 1)$ and $a + 2b < 1$. If there exists a constant $d$ such that for all $s$ large enough we have $\Delta_s - \Delta_{s+1} \leq d(\log s)^a$, then $\lim_{s\to\infty} \Delta_s(\log s)^b/s = 0$. We omit the proof, which is significantly more complicated than that of Theorem 10, since we feel that it is rather a general implication involving such type of subquadratic functions, not providing new insights on the Ramsey function $R(3, s)$ itself.

While we expect that $\lim_{s\to\infty} \Delta_s = \infty$ is true, it can be very difficult to prove. Instead, we propose a weaker statement in Conjecture 11, and we think that it might be provable by constructive methods. This may be feasible by exploiting the techniques used in asymptotic nonprobabilistic lower bound constructions for $R(3, s)$ such as those in [4, 6]. So far such techniques are weaker than the probabilistic methods, but they are more general than the attempts of this paper.

**Conjecture 11.** *There exists integer* $k$ *such that*

$$\lim_{s\to\infty} \sum_{i=0}^{k} \Delta_{s+i} = \infty.$$

Finally, we remark that the constructive methods studied in this paper and previous efforts in similar directions [6, 11, 15] have applications beyond gaining new insights on the growth of $\Delta_s$, like in the study of connectivity and hamiltonicity of Ramsey-critical $(k, s; R(k, s)-1)$-graphs [1, 11, 19], chromatic gaps [11], or in multicolor case, for Shannon capacity of graphs with bounded independence number [18].


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