Multi-Vertex Deletion Graph Reconstruction Numbers

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Abstract

First posed in 1942 by Kelly and Ulam, the *Graph Reconstruction Conjecture* is one of the major open problems in graph theory. While the Graph Reconstruction Conjecture remains open, it has spawned a number of related questions. In the classical vertex graph reconstruction number problem a vertex is deleted in every possible way from a graph G, and then it can be asked how many (both minimum and maximum) of these subgraphs are required to reconstruct G up to isomorphism. This can then be extended to deleting k vertices in every possible way.

Previous computer searches have found the 1-vertex-deletion reconstruction numbers of all graphs of up to 11 vertices. In this paper computed values of k-vertex-deletion reconstruction numbers for all graphs on up to 8 vertices and $k \leq |V(G)| - 2$ are reported, as well as for some k for graphs on 9 vertices. Our data suggested a number of new theorems and conjectures. In particular we pose, as a generalization of the Graph Reconstruction Conjecture, that any graph on $3k$ or more vertices is k-vertex-deletion reconstructible.

1 Introduction

Traditional graph notation (as in [6, 3, 13]) is primarily used in this paper. In all cases graphs are assumed to be simple, undirected, and finite. Furthermore, graphs are considered to be unlabeled, and therefore isomorphic graphs are not distinguished. In the case of common graphs such as cliques (K_n), bipartite/tripartite cliques ($K_{r,s}$/ $K_{r,s,t}$), paths (P_n), and cycles (C_n), the subscripts indicate the number of vertices. The set of all graphs on n vertices is denoted by $\mathcal{G}_n$.

Where more complex graphs need to be labeled, the *graph6* notation (as implemented in Brendan McKay's *nauty* package [15]) is used. This notation uses printable ASCII characters to encode the adjacency matrix of the graph in a compact form. As the adjacency matrix of a graph, and therefore the *graph6* representation, depends on the vertex labeling, the default canonical labeling from *nauty* is used. For instance, the graph $2K_2$ would be written as `` C` `` in *graph6* notation.

In 1942 Kelly and Ulam posed the Graph Reconstruction Conjecture, and it has remained an important open problem to this day.

Definition 1 ([12, 5, 9])**.** $Deck_k(G)$ is the multiset of graphs that results from deleting k vertices in every possible way from the graph G. When a vertex is removed, all edges incident to that vertex are also removed. The elements of a deck are customarily referred to as cards.

Graph Reconstruction Conjecture (Kelly and Ulam, 1942 [10, 12])**.** Any simple finite undirected graph G on 3 or more vertices can be uniquely identified (up to isomorphism) by $Deck_1(G)$.

There are no known counter-examples to this conjecture, and it is widely believed to be true [1, 5, 13, 14]. For some classes of graphs the conjecture has been proven to hold; specifically disconnected graphs, regular graphs, trees, and maximal planar graphs [19, 2, 20, 1]. Through exhaustive computer search it has previously been shown that all graphs of between 3 and 11 vertices [14, 24, 17], and certain classes of graphs of up to 16 vertices [14, 24], are reconstructible. In 1957 Kelly proposed generalizing the Graph Reconstruction Conjecture to deletion of multiple vertices [11].

A graph G is said to be k-vertex reconstructible if it can be uniquely identified (up to isomorphism) from $Deck_k(G)$. More recently the question "if a graph is k-reconstructible, how many of its k-vertex-deleted subgraphs are required to reconstruct it?" has been asked. This takes two forms, the *existential* (or *ally*) *reconstruction number* ($\exists rn_k$), and the *universal* (or *adversarial*) *reconstruction number* ($\forall rn_k$).

Definition 2 ([8, 23, 12, 1])**.** The *existential k-vertex reconstruction number* ($\exists vrn_k$) of a graph G is the cardinality of the smallest $\mathcal{S} \subseteq Deck_k(G)$ that reconstructs G.

Definition 3 ([8, 23, 12, 1])**.** The *universal k-vertex reconstruction number* ($\forall vrn_k$) of a graph G is the smallest number such that all $\mathcal{S} \subseteq Deck_k(G)$ of that cardinality reconstruct G.

If a graph G is not k-vertex reconstructible, then we let $\exists vrn_k(G) = \forall vrn_k(G) = \infty$.

While there is no known efficient way to compute the reconstruction number of a general graph, there are various properties that are known. For example, it has been shown by Bollobás that $\exists vrn_1(G) = \forall vrn_1(G) = 3$ for almost all graphs [4]. There are also a number of classes of graphs which are known to have large (> 3) $\exists vrn_1$, some of which were recently discovered as a result of computations similar to those described in this paper [19, 18, 17].

Less is known about k-vertex reconstruction for $k > 1$. One result from Nỳdl proves that it is possible to construct a graph on $2k$ vertices which is not k-vertex reconstructible for $k \geq 1$ [21, 22] (see also [5]). There has also been some results on the the complexity of decision problems related to k-vertex reconstruction [9], and the 2-vertex reconstructibility of graphs up to 9 vertices [17].

2 Theorems and Conjectures on Reconstruction Numbers

The reconstruction numbers we have computed led to a number of observations. In this section we present theorems generalizing those observations, as well as some conjectures suggested by the data for future investigation.

Theorem 1. *For all $n \geq 3, G \in \mathcal{G}_n$*

$$\exists vrn_{n-2}(G) = \forall vrn_{n-2}(G) = \begin{cases} \binom{n}{n-2} & G \in \mathcal{S} \\ \infty & otherwise \end{cases}$$

where $\mathcal{S} = \{\, nK_1,\ K_n,\ K_2 \cup (n-2)K_1,\ \overline{K_2 \cup (n-2)K_1}\,\}$

Proof. Note that for $n = k + 2$, $Deck_k(G)$ consists of graphs K_2 and $2K_1$, i.e. counting exactly the number of edges in G and providing no other information. Consequently, only graphs reconstructible from their number of edges are k-reconstructible in this case. These are the graphs in $\mathcal{S}$. Observe that for all of them all $\binom{n}{n-2} = \binom{n}{2}$ cards are needed for counting the edges and thus for the reconstruction. $\square$

Lemma 2. *For all $k \geq 1$*

$$\exists vrn_k(G) = \exists vrn_k(\overline{G})$$

$$\forall vrn_k(G) = \forall vrn_k(\overline{G})$$

Proof. If $\mathcal{S}$ is a multiset of graphs, then we let $c(\mathcal{S})$ be the result of taking the complement of each graph in $\mathcal{S}$. Observe that for any graph G, $Deck_k(\overline{G}) = c(Deck_k(G))$ Therefore, a graph H shares subdeck $\mathcal{S}$ with G, iff $\overline{H}$ shares subdeck $c(\mathcal{S})$ with $\overline{G}$. Hence, a subdeck $\mathcal{S}$ uniquely reconstructs G iff $c(\mathcal{S})$ uniquely reconstructs $\overline{G}$, and all subdecks of cardinality s uniquely reconstruct G iff all subdecks of cardinality s uniquely reconstruct $\overline{G}$. □

Theorem 3. *For all $k \geq 1, n \geq k+2, G \in \{nK_1, K_n\}$*

$$\exists vrn_k(G) = \forall vrn_k(G) = \binom{n}{k} - \binom{n-2}{k} + 1$$

Proof. $Deck_k(nK_1)$ consists of $\binom{n}{k}$ cards, each an edgeless graph on $n-k$ vertices. Observe that $Deck_k(K_2 \cup (n-2)K_1)$ has $m = \binom{n}{k} - \binom{n-2}{k}$ edgeless cards (those missing K_2), as there are $\binom{n}{k}$ total cards and $\binom{n-2}{k}$ cards which choose neither vertex of the K_2 subgraph. Similarly, any graph with more than m edgeless cards must be edgeless, because every edge is included in at least one card. Finally, since all cards of nK_1 are the same, $\exists vrn_k(nK_1) = \forall vrn_k(nK_1) = m+1$. By Lemma 2 the same result applies to $K_n = \overline{nK_1}$. □

Theorem 4. *For all $k \geq 1, n \geq k+2, G \in \mathcal{G}_n$*

$$\forall vrn_k(G) \geq \binom{n}{k} - \binom{n-2}{k} + 1$$

Proof. If $G = nK_1$, then $\forall vrn_k(G) = \binom{n}{k} - \binom{n-2}{k} + 1$ by Theorem 3. Otherwise, as in the proof of Theorem 3, for any fixed edge e in G, $Deck_k(G)$ has exactly $m = \binom{n}{k} - \binom{n-2}{k}$ cards obtained by skipping e. Thus G and $G-e$, while nonisomorphic, share a subdeck of m cards. Therefore $m+1$ is a lower bound for $\forall vrn_k(G)$. □

Corollary 5. *For all $k \geq 1, n \geq k+2, G \in \mathcal{G}_n$*

$$\forall vrn_k(G) \geq \forall vrn_k(nK_1)$$

Proof. Follows directly from Theorem 3 and Theorem 4. □

We also pose the following conjectures motivated by data presented in sections 4 and 5. It is easy to check that $\forall vrn_1(K_{1,3}) = 4$, but all further known cases satisfy Conjecture 6.

Conjecture 6. *For all $k \geq 1, n \geq k+3, G \in \{K_{1,n-1}, K_1 \cup K_{n-1}\}$, except for $k = 1, n = 4$*

$$\forall vrn_k(G) = \binom{n}{k} - \binom{n-2}{k} + 1$$

The next conjecture generalizes a well known theorem by Bollobás, which states that almost every graph G has $\exists vrn_1(G) = \forall vrn_1(G) = 3$ [4]. Note that Conjecture 7 only refers to $\exists vrn_k$, as Theorem 4 shows that $\forall vrn_k$ behaves quite differently for $k > 1$.

Conjecture 7. *For all $k \geq 1$, the probability that $\exists vrn_k(G) = 3$ approaches 1 with increasing $|V(G)|$.*

3 Non-Reconstructibility Under k-Vertex-Deletion

While there are no known graphs on more than 3 vertices which are not 1-vertex reconstructible (which is consistent with the Graph Reconstruction Conjecture), this is not true for k-vertex reconstruction for $k > 1$ [21, 22, 17]. Table 1 shows the number of graphs which are not k-vertex reconstructible for values of $|V|$ and k we computed. Clearly $k \geq |V(G)| - 1$ is not of interest, as no graphs are reconstructible for such k. Where there are empty spaces for $n > k+2$, we were not able to compute the result due to prohibitive computation time.[1] Note that Table 1 agrees with Theorem 1, as exactly 4 graphs are computed to be k-vertex reconstructible when $k = |V| - 2$.

		graph order							
		4	5	6	7	8	9	10	11
unique graphs		11	34	156	1044	12346	274668	12005168	1018997864
k	1	0	0	0	0	0	0	0	0
	2	7	4	0	0	0	0		
	3		30	78	20	8	0		
	4			152	854	1937			
	5				1040	11935			
	6					12342	273846		
	7						274664		

Table 1: Number of graphs not k-vertex reconstructible by $|V|$ and k

[1]The results on 9 vertices for $k = 6$ were computed while this paper was in review, and presented only here. The associated data, such as that presented in later sections, is available from the authors.

This data suggests the following definitions and conjectures.

Definition 4. The *k-vertex reconstructible orders* (vro_k) is the set of all n such that all graphs with $|V| = n$ are k-vertex reconstructible — i.e., let $vro_k = \{n : |V(G)| = n \implies G \text{ is } k\text{-vertex reconstructible}\}$.

Definition 5. The *minimal k-vertex reconstructible order* ($\min(vro_k)$) is the minimal value in vro_k. If $vro_k = \emptyset$, then $\min(vro_k) = \infty$.

Conjecture 8. *For all $k \geq 1$, $\min(vro_k) < \infty \quad (vro_k \neq \emptyset)$.*

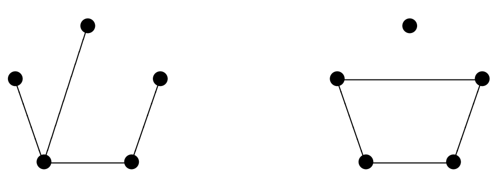
Conjecture 9. *For all $k \geq 1$, $n \geq \min(vro_k) \iff n \in vro_k$.*

Conjecture 10. *For all $k \geq 1$, $\min(vro_k) = 3k$.*

It should be noted that $2k \notin vro_k$ is known due to Nýdl's result in [21, 22]. Proof of Conjecture 9 would be remarkable, as it would also prove the Graph Reconstruction Conjecture. Furthermore, Conjectures 9 and 10 together lead to a generalization of the Graph Reconstruction Conjecture:

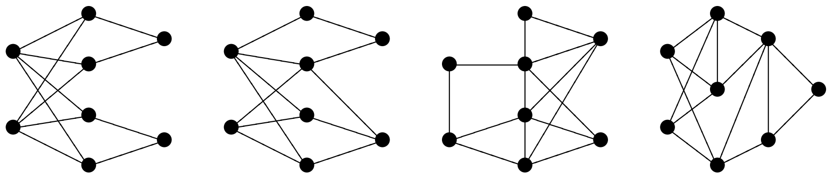
Graph k-Vertex Reconstruction Conjecture. Any simple finite undirected graph G on $3k$ or more vertices can be uniquely identified (up to isomorphism) by $Deck_k(G)$.

The largest non-k-vertex reconstructible graphs are of interest, as there are few of them for the values of k we have computed. It is easy to see that the graphs in Figure 1 have the same $Deck_2$, and by Lemma 2 so do their complements. This result has been previously reported by McMullen in [17, 16]. Analogously, the first two graphs in Figure 2 have the same $Deck_3$. The other two graphs in Figure 2 are of a more interesting variety, as they do not share a $Deck_3$ with each other, but each with its own complement.



(A) `D@s` (B) `DIK`

Figure 1: Graphs on 5 vertices which, along with their complements, are not 2-vertex reconstructible. The sets which share the same $Deck_2$ are: $\{A, B\}, \{\overline{A}, \overline{B}\}$.



(A) GSP@xw (B) G_Kta[(C) GG_}p{ (D) G_[pm[

Figure 2: Graphs on 8 vertices which, along with their complements, are not 3-vertex reconstructible. The sets which share the same $Deck_3$ are: $\{A, B\}, \{\overline{A}, \overline{B}\}$, $\{C, \overline{C}\}, \{D, \overline{D}\}$

4 Existential k-Vertex-Deletion Reconstruction Numbers

This section presents values of $\exists vrn_k$ we have computed for varying k. Table 2 shows the distribution of $\exists vrn_1$ according to number of vertices for all graphs up to 11 vertices, a result we previously reported in [24]. Note that $|V| = 3$ is not shown as Theorem 1 gives us exact values.

Tables 3, 5, 7, and 9 show the same information for k-vertex-deletion for $2 \leq k \leq 5$. We have computed $\exists vrn_{|V|-2}$ for all graphs on up to 9 vertices, and the results do match Theorem 1, so we only display results for $|V| \geq k + 3$. For $2 \leq k \leq 5$ we list those graphs which (along with their complements) have maximal $\exists vrn_k$ for each order in Tables 4, 6, 8, and 10. Graphs with $\exists vrn_1 > 3$ and $|V| \leq 11$ have previously been described [19, 18, 17, 24], and are not repeated here. It is clear that it is more common for $\exists vrn_k$ to be greater than 3 when $k > 1$. However, an obvious pattern appears whereby the ratio of graphs with $\exists vrn_k(G) = 3$ increases as $|V(G)|$ increases, regardless the value of k. This pattern led to the formulation of Conjecture 7 in section 2.

		graph order							
		4	5	6	7	8	9	10	11
unique graphs		11	34	156	1044	12346	274668	12005168	1018997864
$\exists vrn_1$	3	8	34	150	1044	12334	274666	12005156	1018997864
	4	3		4		8		6	
	5			2		2	2	4	
	6					2			
	7							2	

Table 2: Counts of $\exists vrn_1$ by number of vertices

		graph order				
		5	6	7	8	9
unique graphs		34	156	1044	12346	274668
not reconstructible		4	0	0	0	0
$\exists vrn_2$	3		8	240	9592	270869
	4	2	30	396	2464	3454
	5		34	216	216	230
	6	4	30	106	36	50
	7	8	32	44	18	20
	8	9	16	20	8	16
	9	7	2	10	2	5
	10		2	4	4	12
	11		2	2	4	4
	12			6		2
	13					
	14				2	2
	15					
	16					4

Table 3: Counts of $\exists vrn_2$ by number of vertices

graph		$\|V\|$	$\|E\|$	$\exists vrn_2$	$\forall vrn_2$
$2K_2 \cup K_1$	`DGC`	5	2	9	10
$P_4 \cup K_1$	`DAK`	5	3	9	10
P_5	`DDW`	5	4	9	10
C_5	`DqK`	5	5	9	9
$3K_2$	``E`?G``	6	3	11	13
$7K_1$	`F????`	7	0	12	12
$3K_2 \cup K_1$	`FGC?G`	7	3	12	16
$K_4 \cup K_3$	`FwCWw`	7	9	12	14
$8K_1$	`G?????`	8	0	14	14
$9K_1$	`H??????`	9	0	16	16
$K_5 \cup 6K_4$	`H~?GW[N`	9	16	16	22

Table 4: Graphs which, along with their complements, have maximal $\exists vrn_2$ for each order

		graph order			
		6	7	8	9
unique graphs		156	1044	12346	274668
not reconstructible		78	20	8	0
$\exists vrn_3$	3				2760
	4			128	45713
	5		10	652	145271
	6		12	1738	62156
	7		24	2290	14434
	8	2	66	2285	3018
	9	2	90	1874	678
	10	4	126	1216	244
	11		88	755	160
	12	2	96	490	68
	13	8	70	304	46
	14	2	76	207	34
	15	10	54	152	26
	16	8	66	72	20
	17	14	74	40	8
	18	22	54	38	2
	19	4	62	36	8
	20		30	5	2
	21		14	16	2
	22		6	6	4
	23			6	2
	24		4	6	4
	25			2	
	26		2	4	
	27				
	28			4	
	29			4	
	30			2	
	31				2
	32				
	33			2	
	36				4
	37			2	
	41			2	
	50				2

Table 5: Counts of $\exists vrn_3$ by number of vertices

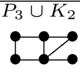
graph		$\|V\|$	$\|E\|$	$\exists vrn_3$	$\forall vrn_3$
$P_3 \cup K_2 \cup K_1$	`E?D_`	6	3	19	19
	`EANg`	6	7	19	19
$7K_1$	`F????`	7	0	26	26
$2K_4$	`G~?GW[`	8	12	41	49
$9K_1$	`H??????`	9	0	50	50

Table 6: Graphs which, along with their complements, have maximal $\exists vrn_3$ for each order

		graph order	
		7	8
unique graphs		1044	12346
not reconstructible		854	1937
$\exists vrn_4$	7		6
	8		6
	9		10
	10		21
	11		8
	12		16
	13		48
	14		66
	15	4	100
	16	6	170
	17	2	193
	18	2	212
	19	2	346
	20	2	440
	21	4	368
	22		310
	23	2	318
	24		365
	25	14	375
	26	2	436
	27	6	322
	28	22	420
	29	8	460
	30	16	488
	31	30	452
	32	22	434
	33	44	442
	34	2	442
	35		450
	36		354
	37		370
	38		351
	39		403
	40		300
	41		304
	42		212
	43		169
	44		70
	45		58
	46		36
	47		22
	48		20
	49		6
	50		4
	51		2
	52		2
	56		2

Table 7: Counts of $\exists vrn_4$ by number of vertices

graph		$\|V\|$	$\|E\|$	$\exists vrn_4$	$\forall vrn_4$
$(K_{2,4} - e) \cup K_1$	`F??zo`	7	7	34	34
$8K_1$	`G?????`	8	0	56	56

Table 8: Graphs which, along with their complements, have maximal $\exists vrn_4$ for each order

		$\|V\|$
		8
unique graphs		12346
not reconstructible		11935
$\exists vrn_5$	23	2
	24	4
	25	4
	26	4
	27	
	28	2
	29	6
	30	4
	31	2
	32	2
	33	
	34	4
	35	2
	36	2
	37	2
	38	4
	39	4
	40	6
	41	2
	42	8
	43	8
	44	8
	45	8
	46	12
	47	32
	48	10
	49	28
	50	30
	51	24
	52	56
	53	66
	54	65

Table 9: Counts of $\exists vrn_5$ for $|V| = 8$

graph	$\|V\|$	$\|E\|$	$\exists vrn_5$	$\forall vrn_5$
`G@?G?C`	8	3	54	55
`G?C?J?`	8	4	54	55
``G`?G?C``	8	4	54	55
`G?C?G[`	8	5	54	55
`G???z[`	8	8	54	55
`G???~K`	8	8	54	55
`G?C?N[`	8	8	54	55
`G?GGg{`	8	8	54	55
`G??@}w`	8	9	54	55
`G??Hb{`	8	9	54	55
`G??Hfw`	8	9	54	55
`G??O^s`	8	9	54	55
`G?CZFC`	8	9	54	55
`G??Ix{`	8	10	54	55
`G??gx{`	8	10	54	55
`GGC?N{`	8	10	54	55
`G??Yx{`	8	11	54	55
`G??gz{`	8	11	54	55
`G?@@~s`	8	11	54	55
`G?AJjw`	8	11	54	55
`G_?Dzw`	8	11	54	55
`G_?gx{`	8	11	54	55
`G??zvo`	8	12	54	55
`G?CNnW`	8	12	54	55
`G?LLng`	8	13	54	55
`G@NEJs`	8	13	54	55
`G@hYtK`	8	13	54	55
`G_GXx{`	8	13	54	55
``G`GWx{``	8	13	54	55
`G?@zvs`	8	14	54	55
`G?G\z{`	8	14	54	55
`G_Azvo`	8	14	54	55
``G`iayw``	8	14	54	55

Table 10: Graphs which, along with their complements, have maximal $\exists vrn_5$ for $|V| = 8$

5 Universal k-Vertex-Deletion Reconstruction Numbers

This section presents values of $\forall vrn_k$ we have computed for varying values of k, analogously to Section 4. Table 11 shows the distribution of $\forall vrn_1$ according to number of vertices for all graphs up to 11 vertices, a result we previously reported in [24]. As before $|V| = 3$ is not shown as Theorem 1 gives us exact values.

Tables 12, 14, 16, and 18 show the same information for k-vertex-deletion for $2 \leq k \leq 5$. We have computed $\forall vrn_{|V|-2}$ for a graphs on up to 9 vertices, and the results do match Theorem 1, so we only display results for $|V| \geq k+3$. For $2 \leq k \leq 5$ we list those graphs which (along with their complements) have minimal $\forall vrn_k$ for each order in Tables 13, 15, 17, and 19. Since almost all graphs have the minimal $\forall vrn_1 = 3$ [4] we do not give a listing of them. As some of the graphs listed in Table 13 are too complex to give a succinct description, they are shown in Figure 3. Note that those graphs in Figure 3 appear related, but to not make up an obvious family.

Another interesting pattern that emerges involves the maximal $\forall vrn_k$. By Theorem 1 the maximal $\forall vrn_k$ of graphs on $k+2$ vertices is $\binom{|V|}{k}$. For slightly larger graphs it appears to be $\binom{|V|}{k} - c$ for small values of c. For example, our data shows that the maximal $\forall vrn_2$ for $|V| = 6$, $\forall vrn_3$ for $|V| \in \{6, 7\}$, $\forall vrn_4$ for $|V| \in \{7, 8\}$, and $\forall vrn_5$ for $|V| = 8$ are all $\binom{|V|}{k} - 1$.

		graph order							
		4	5	6	7	8	9	10	11
unique graphs		11	34	156	1044	12346	274668	12005168	1018997864
$\forall vrn_1$	3	2	7	8	16	266	45186	6054148	815604300
	4	9	19	56	496	8208	199247	5637886	199382868
	5		8	90	520	3584	28781	301530	3922130
	6			2	12	284	1434	10686	83730
	7					4	20	914	4824
	8							4	12

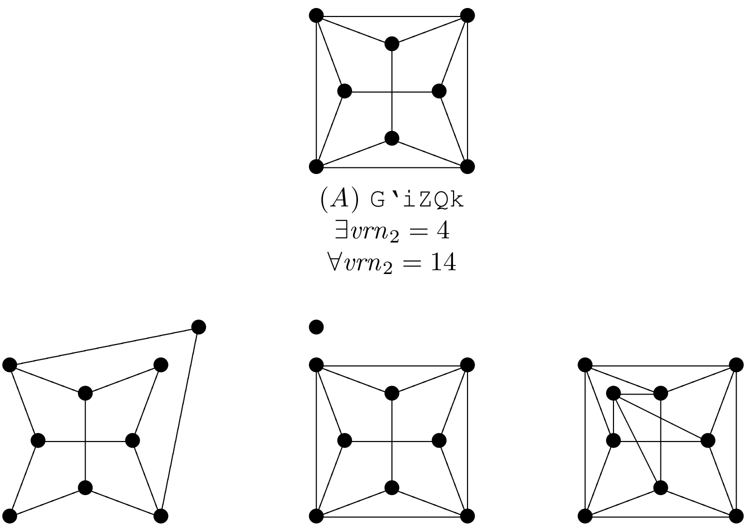
Table 11: Counts of $\forall vrn_1$ by number of vertices

		graph order				
		5	6	7	8	9
unique graphs		34	156	1044	12346	274668
not reconstructible		4	0	0	0	0
$\forall vrn_2$	8	6				
	9	9				
	10	15	6			
	11		2			
	12		4	4		
	13		98	2		
	14		46	14	5	
	15			76	4	
	16			216	36	9
	17			532	111	271
	18			172	1020	3704
	19			28	2820	14270
	20				3598	21982
	21				3212	60137
	22				1254	79798
	23				248	48632
	24				32	20508
	25				6	17347
	26					5772
	27					1826
	28					316
	29					92
	30					4

Table 12: Counts of $\forall vrn_2$ by number of vertices

graph		$\|V\|$	$\|E\|$	$\exists vrn_2$	$\forall vrn_2$
$5K_1$	`D??`	5	0	8	8
$K_{1,4}$	`D?{`	5	4	7	8
$K_3 \cup K_2$	``D`K``	5	4	8	8
$6K_1$	`E???`	6	0	10	10
$K_{1,5}$	`E?Bw`	6	5	8	10
$K_4 \cup 2K_1$	`E@Kw`	6	6	5	10
$7K_1$	`F????`	7	0	12	12
$K_{1,6}$	`F??Fw`	7	6	9	12
$8K_1$	`G?????`	8	0	14	14
$K_{1,7}$	`G???F{`	8	7	10	14
	``G`iZQk``	8	14	4	14
$9K_1$	`H??????`	9	0	16	16
$K_{1,8}$	`H????B~`	9	8	11	16
	``HC`PX`H``	9	12	4	16
	`HGDQXgj`	9	14	3	16
	`H{dQXgj`	9	18	9	16

Table 13: Graphs which, along with their complements, have minimal $\forall vrn_2$ for each order



(A) ``G`iZQk``
$\exists vrn_2 = 4$
$\forall vrn_2 = 14$

(B) ``HC`PX`H``
$\exists vrn_2 = 4$
$\forall vrn_2 = 16$

(C) `HGDQXgj`
$\exists vrn_2 = 3$
$\forall vrn_2 = 16$

(D) `H{dQXgj`
$\exists vrn_2 = 9$
$\forall vrn_2 = 16$

Figure 3: Complex graphs from Table 13

		graph order			
		6	7	8	9
unique graphs		156	1044	12346	274668
not reconstructible		78	20	8	0
$\forall vrn_3$	17	4			
	18	6			
	19	68			
	26		6		
	27				
	28		4		
	29		4		
	30		38		
	31		88		
	32		400		
	33		342		
	34		142		
	37			8	
	38				
	39			4	
	40			3	
	41			16	
	42			6	
	43			51	
	44			76	
	45			263	
	46			532	
	47			1282	
	48			2451	
	49			3902	
	50			2602	5
	51			840	
	52			118	2
	53			96	8
	54			88	4
	55				75
	56				98
	57				157
	58				242
	59				360
	60				1940
	61				3798
	62				6426
	63				11409
	64				21181
	65				32518
	66				42127
	67				46011
	68				38908
	69				30087
	70				18289
	71				10642
	72				5843
	73				2984
	74				1216
	75				224
	76				64
	77				46
	78				
	79				4

Table 14: Counts of $\forall vrn_3$ by number of vertices

graph		$|V|$	$|E|$	$\exists vrn_3$	$\forall vrn_3$
$6K_1$	`E???`	6	0	17	17
$K_{1,5}$	`E?Bw`	6	5	15	17
$7K_1$	`F????`	7	0	26	26
$K_{1,6}$	`F??Fw`	7	6	21	26
$K_5 \cup 2K_1$	`F@Kxw`	7	10	10	26
$8K_1$	`G?????`	8	0	37	37
$K_{1,7}$	`G???F{`	8	7	28	37
$K_{2,6}$	`G??F~w`	8	12	14	37
$K_{1,1,6}$	`G??F~{`	8	13	14	37
$9K_1$	`H??????`	9	0	50	50
$K_{1,8}$	`H????B~`	9	8	36	50
	`H{dQXgj`	9	18	6	50

Table 15: Graphs which, along with their complements, have minimal $\forall vrn_3$ for each order

		graph order	
		7	8
unique graphs		1044	12346
not reconstructible		854	1937
$\forall vrn_4$	31	8	
	32	6	
	33	16	
	34	160	
	56		8
	57		2
	60		4
	61		14
	62		22
	63		98
	64		214
	65		548
	66		1065
	67		3062
	68		3362
	69		2010

Table 16: Counts of $\forall vrn_4$ by number of vertices

graph		$|V|$	$|E|$	$\exists vrn_4$	$\forall vrn_4$
$7K_1$	`F????`	7	0	31	31
$K_{1,6}$	`F??Fw`	7	6	27	31
$K_4 \cup K_3$	`FwCWw`	7	9	28	31
$K_{2,5}$	`F?B~o`	7	10	29	31
$8K_1$	`G?????`	8	0	56	56
$K_{1,7}$	`G???F{`	8	7	47	56
$K_{2,6}$	`G??F~w`	8	12	21	56
$K_{1,1,6}$	`G??F~{`	8	13	21	56

Table 17: Graphs which, along with their complements, have minimal $\forall vrn_4$ for each order

		$\lvert V\rvert$
		8
unique graphs		12346
not reconstructible		11935
$\forall vrn_5$	51	6
	52	6
	53	18
	54	24
	55	357

Table 18: Counts of $\forall vrn_5$ for $|V| = 8$

graph		$\lvert V\rvert$	$\lvert E\rvert$	$\exists vrn_5$	$\forall vrn_5$
$8K_1$	`G?????`	8	0	51	51
$K_{1,7}$	`G???F{`	8	7	45	51
$K_{2,6}$	`G??F~w`	8	12	47	51

Table 19: Graphs which, along with their complements, have minimal $\forall vrn_5$ for $|V| = 8$

6 Algorithm

All the results presented in previous sections were obtained by using the same basic algorithms which were described in [24]. After introducing some notation on multisets (of graphs), this section describes the main algorithm.

Definition 6.

(a) $m(S;\, x)$ is the multiplicity of an element x in a multiset $\mathcal{S}$ (the number of times x appears in $\mathcal{S}$).

(b) $|\mathcal{S}| = \sum_{x\in\mathcal{S}} m(\mathcal{S};\, x)$ is the cardinality of a multiset $\mathcal{S}$.

(c) $\mathbb{B}(\mathcal{S};\, q) = \{x \mid m(S;\, x) \geq q\}$ is the set of elements in $\mathcal{S}$ with multiplicity at least q. If q is omitted, then it is presumed to be 1, giving the basis set of $\mathcal{S}$.

The intersection ($\bigcap$) and union ($\bigcup$) of multisets preserves the minimal and maximal multiplicity of matching elements, while the additive union ($\biguplus$) sums the multiplicities of matching elements. Thus we have:

- $m(\mathcal{S}_1 \bigcap \mathcal{S}_2;\, x) = \min(m(\mathcal{S}_1;\, x), m(\mathcal{S}_2;\, x))$
- $m(\mathcal{S}_1 \bigcup \mathcal{S}_2;\, x) = \max(m(\mathcal{S}_1;\, x), m(\mathcal{S}_2;\, x))$
- $m(\mathcal{S}_1 \biguplus \mathcal{S}_2;\, x) = m(\mathcal{S}_1;\, x) + m(\mathcal{S}_2;\, x)$

In the following, a set will be considered to be a special case of multiset, where the multiplicity of all elements is one.

To determine both universal and existential reconstruction numbers the same primitive question is asked: "can a given subdeck $\mathcal{S}$ reconstruct G?" In order for $\mathcal{S}$ to not reconstruct G there must be another graph H which also has $\mathcal{S}$ as a subdeck. Therefore, in order to answer the question, either an example of a graph which shares the same subdeck must be found, or it must be proven that no such graph exists. We answer that question by computational search.

In order to narrow down the search space of graphs which may share a given subdeck, only graphs which share at least one card with G are considered. An expedient way of obtaining that search space is to perform the inverse operation to $Deck_k$ for each $C \in Deck_k(G)$.

Definition 7. $Extensions_k(F)$ is the set of non-isomorphic graphs that results from adding k vertices to the graph F, and adding edges incident to the new vertices in every possible way.

The following algorithm, inspired by that used by Brian McMullen [17, 16], was used to compute the reconstruction results presented earlier:

1. $\mathcal{D}_G \leftarrow Deck_k(G)$
2. for each $C \in \mathcal{D}_G$, compute multiset $\mathcal{H}_C$:
   (a) set the basis set of $\mathcal{H}_C$ to be $Extensions_k(C) - G$
   (b) for each $H \in \mathcal{H}_C$ let $m(\mathcal{H}_C; H) \leftarrow \min(\ m(Deck_k(H);\ C),\ m(\mathcal{D}_G;\ C)\ )$
3. $\mathcal{H} \leftarrow \biguplus_{C \in \mathcal{D}_G} \mathcal{H}_C$
4. let $\forall vrn_k(G) \leftarrow 1 + \max(m(\mathcal{H};\ H) :\ \ H \in \mathcal{H})$
5. let $\exists vrn_k(G) \leftarrow \min(\ |\mathcal{S}| :\ \ (\mathcal{S} \subseteq \mathcal{D}_G) \wedge (\bigcap_{C \in \mathcal{S}} \mathbb{B}(\mathcal{H}_C;\ m(\mathcal{S};\ C)) \ = \emptyset)\ )$

The multisets labeled $\mathcal{H}_C$ are constructed such that each $H \in \mathcal{H}_C$ has a multiplicity equal to the number of times C is shared in the decks of G and H. The multiset $\mathcal{H}$ then has multiplicities of each $H \in \mathcal{H}$ equal to the total number of cards H shares with G.

It is important to note that since isomorphic graphs are considered equivalent, a common implicit operation in this algorithm is the test of isomorphism. This is accomplished by use of the canonical labeling function in Brendan McKay's *nauty* [15] package. Each graph is canonically labeled as it is generated, and thereafter is simply tested for equality with others.

As canonical labeling itself is an expensive operation, it is beneficial to reduce the number of times it must be performed. The structural differences with the algorithm used in [16] and [17] are designed to reduce the number of canonical labelings that are required. By taking advantage of the fact that $H \in Extensions_k(C) \implies C \in Deck_k(H)$, we can see that $m(Deck_k(G);\ C) = 1 \implies m(\mathcal{H}_c;\ H) = 1$ in step 2b without performing any further calculations. To further optimize cases where $m(Deck_k(G);\ C) > 1$, it can be noted that computing $m(Deck_k(H);\ C)$ only requires the inspection of those graphs in $Deck_k(H)$ that have the same number of edges as C.

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