Bounds on Some Ramsey Numbers Involving Quadrilateral †

Xiaodong Xu
Guangxi Academy of Sciences
Nanning,Guangxi 530007, China
xxdmaths@sina.com

Zehui Shao
Department of Control Science and Engineering
Huazhong University of Science and Technology
Wuhan 430074, China
kidszh_mail@163.com

Stanisław P. Radziszowski
Department of Computer Science
Rochester Institute of Technology
Rochester, NY 14623, USA
spr@cs.rit.edu

Abstract. For graphs $G_1, G_2, \cdots, G_m$, the Ramsey number $R(G_1, G_2, \cdots, G_m)$ is defined to be the smallest integer n such that any m-coloring of the edges of the complete graph K_n must include a monochromatic G_i in color i, for some i. In this note we establish several lower and upper bounds for some Ramsey numbers involving quadrilateral C_4, including $R(C_4, K_9) \leq 32$, $19 \leq R(C_4, C_4, K_4) \leq 22$, $31 \leq R(C_4, C_4, C_4, K_4) \leq 50$, $52 \leq R(C_4, K_4, K_4) \leq 72$, $42 \leq R(C_4, C_4, K_3, K_4) \leq 76$, and $87 \leq R(C_4, C_4, K_4, K_4) \leq 179$.

†Supported by the National Natural Science Foundation of China (Grant Nos. 60533010, 60703047 and 60563008), the Basic Research Fund of Guangxi Academy of Sciences (080414), the Program for New Century Excellent Talents in University (NCET-05-0612) and the Chenguang Program of Wuhan (200750731262).

1 Introduction and Overview

We consider only graphs without multiple edges or loops. If $G = (V, E)$ is a graph, then the set of vertices of G is denoted by $V(G)$, the set of edges by $E(G)$, and their cardinalities by $|V(G)|$ and $|E(G)|$, respectively. We denote by $\delta(G)$ the minimum degree of G, $N(v)$ the neighborhood of vertex v and $N[v] = \{v\} \cup N(v)$. If the degree of each vertex of G is k, then G is called a k-regular graph. The graph C_n is a cycle of length n and K_n a complete graph of order n. Define $t(n)$ to be the maximum number of edges in any graph of order n not containing C_4.

For graphs $G_1, G_2, \cdots, G_m$, the Ramsey number $R(G_1, G_2, \cdots, G_m)$ is defined to be the least positive integer n such that every m-coloring of the edges of the complete graph K_n contains a monochromatic subgraph isomorphic to G_i whose all edges have color i, for some i, $1 \leq i \leq m$. A coloring of the edges of complete graph with m colors is called a $(G_1, G_2, \cdots, G_m)$-coloring (or graph for $m = 2$), if it does not contain a subgraph isomorphic to G_i whose all edges are colored with color i, for each i. A $(G_1, G_2, \cdots, G_m)$-coloring (graph) of K_n is denoted as $(G_1, G_2, \cdots, G_m; n)$. For other graph theory concepts refer to [5] and [4].

In this note we establish several lower and upper bounds for some Ramsey numbers involving quadrilateral C_4. The lower bounds are obtained by an improvement to Bevan's construction [2] or an explicit coloring. Several concrete upper bounds are obtained by using general properties of the function $t(n)$ first derived by Irving [8] and later enhanced by others [6, 14]. Table 1 summarizes all known values and bounds on Ramsey numbers of the form $R(C_4, G_1, G_2)$ and $R(C_4, C_4, G_1, G_2)$ where both of G_1 and G_2 are one of the graphs C_4, $C_3(= K_3)$ or K_4. The bounds in Table 1 without references are obtained in the next two sections. For completeness we note that $R(C_4, C_4) = 6$, $R(C_4, C_3) = R(C_4, K_3) = 7$, and $R(C_4, K_4) = 10$ (cf. [9]). For known values and bounds on $R(C_4, K_n)$ for higher n see [10, 9]. In section 3 we derive a new upper bound $R(C_4, K_9) \leq 32$, which in turn easily implies $R(C_4, K_{10}) \leq 39$.

In the sequel we don't discuss asymptotic problems associated with such types of Ramsey numbers. Let us only mention that several interesting results were obtained by Alon and Rödl [1], including $R(C_4, C_4, K_m) = \Theta(m^2 poly(\log m))$ and $R(C_4, C_4, C_4, K_m) = \Theta(m^2 / \log^2 m)$. For the discussion of asymptotics in the two-color case see [10]. For many other references to multicolor cases consult the results and citations in [1, 9].

Ramsey number parameters	value/ bounds	reference
C_4, C_4, C_4	11	[3]
C_4, C_4, C_3	12	[11]
C_4, C_4, K_4	19-22	
C_4, C_3, C_3	17	[7]
C_4, C_3, K_4	25-32	
C_4, K_4, K_4	52-72	
C_4, C_4, C_4, C_4	18	[12]
C_4, C_4, C_4, C_3	21-27	[13]
C_4, C_4, C_4, K_4	31-50	
C_4, C_4, C_3, C_3	28-36	[13]
C_4, C_4, C_3, K_4	42-76	
C_4, C_4, K_4, K_4	87-179	

Table 1. $R(C_4, G_1, G_2)$ and $R(C_4, C_4, G_1, G_2)$, values and bounds for $G_1, G_2 \in \{C_4, C_3, K_4\}$.

2 Constructive Lower Bounds

Several general lower bound constructions for multicolor Ramsey numbers avoiding complete and other graphs are listed in the survey [9]. We first cite as Theorem 1, and then extend it to Theorem 2, a version derived by Bevan in 2002 [2].

Theorem 1 *For arbitrary connected graphs $G_1, G_2, \cdots, G_r$, we have* $R(G_1, \cdots, G_r, K_{k_1}, \cdots, K_{k_s}) \geq (R(G_1, \cdots, G_r) - 1)(R(k_1, \cdots, k_s) - 1) + 1.$

Theorem 1 and lower bounds listed in [9] easily imply the following.

Corollary 1
(1) $R(C_4, K_3, K_4) \geq 25 \quad (= 3 \cdot 8 + 1)$,
(2) $R(C_4, K_4, K_4) \geq 52 \quad (= 3 \cdot 17 + 1)$,
(3) $R(C_4, C_4, C_4, K_4) \geq 31 \quad (= 10 \cdot 3 + 1)$.

We can improve on the Bevan's construction at least in some cases as follows.

Theorem 2
(1) $87 \le R(C_4, C_4, K_4, K_4)$,
(2) $42 \le R(C_4, C_4, K_3, K_4)$.

Proof. (1) Let G be the Paley graph of order 17 with the vertex set $V(G) = Z_{17} = \{0, 1, \cdots, 16\}$. The edge between i and j is in color 3 iff either $|i-j|$ or $17 - |i-j|$ is in the set $S = \{1, 2, 4, 8\}$, and other edges are in color 4. Up to renaming of colors, this is the well known self-complementary unique $(4, 4; 17)$-coloring. Note that every triangle in color 3 uses at least one edge yielding value 1 or 4 in S.

Consider a graph H (isomorphic to C_5) with the vertices $V(H) = \{v_j | 1 \le j \le 5\}$, the edges $\{(v_j, v_{j+1}) | 1 \le j \le 4\} \cup \{(v_1, v_5)\}$ in color 1, and the other edges in color 2. First, we construct the product graph $G[H]$ with vertices $V(G) \times V(H)$, so its order is 85. For different $j_1, j_2 \in \{1, 2, 3, 4, 5\}$ and $i \in \{0, 1, \cdots, 16\}$, the edge between (i, v_{j_1}) and (i, v_{j_2}) is in the same color as that of the edge (v_{j_1}, v_{j_2}) in H. For any $j_1, j_2 \in \{1, 2, 3, 4, 5\}$ and different $i_1, i_2 \in \{0, 1, \cdots, 16\}$, the edge between (i_1, v_{j_1}) and (i_2, v_{j_2}) is in the same color as that of (i_1, i_2) in G. This coloring of K_{85} is the same as one in the proof of Theorem 1, and it is not difficult to see that it is a $(C_4, C_4, K_4, K_4; 85)$-coloring.

We extend $G[H]$ by a new vertex w. For each $i \in Z_{17}$, we color the edge between w and (i, v_1) with color 3. For each $i \in Z_{17}$ and $j \in \{2, 3, 4, 5\}$, we color the edge between w and (i, v_j) with one of the colors 1 or 2, which is different from the color of (v_1, v_j) in H. Note that no C_4's in color 1 or 2 are created, however now some monochromatic K_4's in color 3 containing w are present. Thus, finally, for all $i \in Z_{17}$, we recolor the edges between (i, v_1) and $(i+1, v_1)$ with color 1, and recolor the edges between (i, v_1) and $(i+4, v_1)$ with color 2. One can easily see that this does not create C_4's in colors 1 or 2, but eliminates monochromatic K_4's in color 3 containing w (since, as observed above, every triangle in color 3 in G has an edge yielding difference 1 or 4 in S). Hence $R(C_4, C_4, K_4, K_4) \ge 87$.

(2) We use the same method as in part (1). Graph G is the cyclic $(3, 4; 8)$-graph with the arc set $S = \{1, 4\}$, while H is the same as before. This leads to the construction of a $(C_4, C_4, K_3, K_4; 40)$-coloring. One can easily check that no forbidden monochromatic subgraphs are created after adding new vertex w and simple recoloring of the edges. Thus we have $R(C_4, C_4, K_3, K_4) \ge 42$. $\square$

Theorem 3 $19 \leq R(C_4, C_4, K_4)$.

Proof. The unique graph G of order 18 containing no C_4 with the maximum of 39 edges was found in [6]. Consider the edges of G to be of color 1. Using the computer we colored the complement of G using colors 2 and 3, so that there is no C_4 in color 2 and no K_4 in color 3. The resulting $(C_4, C_4, K_4; 18)$-coloring, proving that $19 \leq R(C_4, C_4, K_4)$, is shown in Figure 1.

1	0	2	2	2	2	3	3	3	3	1	3	2	1	1	3	3	3	3
2	2	0	2	3	3	3	2	2	3	2	1	3	3	3	1	3	1	3
3	2	2	0	3	3	2	3	3	2	3	2	1	3	3	3	1	3	1
4	2	3	3	0	1	3	3	3	3	1	3	2	3	3	1	1	2	2
5	2	3	3	1	0	3	3	3	3	1	3	3	3	3	2	2	1	1
6	3	3	2	3	3	0	3	3	1	3	2	1	1	2	1	3	2	3
7	3	2	3	3	3	3	0	1	3	2	1	3	2	1	3	1	3	2
8	3	2	3	3	3	3	1	0	3	3	1	3	1	2	3	2	3	1
9	3	3	2	3	3	1	3	3	0	3	3	1	2	1	2	3	1	3
10	1	2	3	1	1	3	2	3	3	0	1	1	3	3	2	3	2	3
11	3	1	2	3	3	2	1	1	3	1	0	1	3	3	3	2	3	2
12	2	3	1	2	3	1	3	3	1	1	1	0	2	2	3	3	3	3
13	1	3	3	3	3	1	2	1	2	3	3	2	0	2	1	3	3	1
14	1	3	3	3	3	2	1	2	1	3	3	2	2	0	3	1	1	3
15	3	1	3	1	2	1	3	3	2	2	3	3	1	3	0	1	2	3
16	3	3	1	1	2	3	1	2	3	3	2	3	3	1	1	0	3	2
17	3	1	3	2	1	2	3	3	1	2	3	3	3	1	2	3	0	1
18	3	3	1	2	1	3	2	1	3	3	2	3	1	3	3	2	1	0

Figure 1. Matrix of a $(C_4, C_4, K_4; 18)$-coloring.

$\square$

3 Upper Bounds

The main tool we use in this section for establishing some upper bounds on Ramsey numbers involving C_4 are the known values and upper bounds on $t(n)$ (the maximum number of edges in a C_4-free graph on n vertices). Lemma 1 below summarizes the facts about $t(n)$ studied by various authors [8, 4, 6, 14] which we will need. The exact values of $t(n)$ up to 21 were obtained in [6], and the list was extended to up to 31 in [14].

Lemma 1

(1) *For any n-vertex C_4-free graph G, $n > 3$,*
$|E(G)| \leq t(n) < \frac{1}{4}n(1+\sqrt{4n-3})$ *and* $\delta(G) < \frac{1}{2}(1+\sqrt{4n-3})$.

(2) $t(22) = 52$, $t(29) = 80$, $t(31) = 90$.

Theorem 4

(1) $R(C_4, K_9) \leq 32$, (2) $R(C_4, K_{10}) \leq 39$.

Proof. (1) Suppose G is a $(C_4, K_9; 32)$-graph. From $R(C_4, K_8) = 26$ [10] we see that $\delta(G) \geq 6$. By Lemma 1.1 $\delta(G) \leq 6$, it follows $\delta(G) = 6$. And since $t(31) = 90$, we have $|E(G)| \leq 96$. Hence $|E(G)| = 96$ and G is a 6-regular graph. If G has a triangle xyz, then one can easily count that $V(G) - \{x, y, z\}$ induces in G a graph on $96 - 3 - 12 = 81$ edges, which by Lemma 1.2 is impossible since $t(29) = 80$. Thus G has no triangles. For any vertex x, there are $30 = 6 \times 5$ edges between $X = N[x]$ and $Y = V(G) - X$. Note that Y has 25 vertices, hence for some y in Y, y is connected to at least two vertices s, t in $N(x)$. Thus $xsyt$ forms a C_4, a contradiction, and we have $R(C_4, K_9) \leq 32$.

(2) Suppose G is a $(C_4, K_{10}; 39)$-graph. Now $R(C_4, K_9) \leq 32$ implies that $\delta(G) \geq 7$ which contradicts Lemma 1.1. Therefore, $R(C_4, K_{10}) \leq 39$. □

The best known lower bounds for the cases considered in Theorem 4 are $30 \leq R(C_4, K_9)$ and $34 \leq R(C_4, K_{10})$ [10].

Theorem 5

(1) $R(C_4, C_4, K_4) \leq 22$,
(2) $R(C_4, K_3, K_4) \leq 32$,
(3) $R(C_4, K_4, K_4) \leq 72$,
(4) $R(C_4, C_4, C_4, K_4) \leq 50$,
(5) $R(C_4, C_4, K_3, K_4) \leq 76$,
(6) $R(C_4, C_4, K_4, K_4) \leq 179$.

Proof.
(1) Suppose G is a $(C_4, C_4, K_4; 22)$-graph. Since $R(C_4, C_4, K_3) = 12$ [11] then for each $v \in V(G)$, the number of edges of color 3 incident to v is at most 11, and their total number is at most $11 \cdot 22/2 = 121$. Since $t(22) = 52$, the number of colored edges in G is at most $2t(22) + 121 = 225$, which is less than $|E(K_{22})| = 231$, a contradiction.

(2) $R(C_4, K_3, K_4) \leq R(C_4, K_9) \leq 32$ since $R(K_3, K_4) = 9$.

(3) Suppose G is a $(C_4, K_4, K_4; 72)$-graph. $R(C_4, K_3, K_4) \leq 32$ implies that for each vertex $v \in G$ the number of edges of colors 2 and 3 incident to v are both at most 31. By Lemma 1.1 $t(72) \leq 321$, hence the number of colored edges in G is at most $321 + 72 \cdot 31 = 2553$, which is less than $|E(K_{72})| = 2556$, a contradiction.

(4) In order to obtain $R(C_4, C_4, C_4, K_4) \leq 50$, proceed similarly as in (1) and (3) using $R(C_4, C_4, C_4, K_3) \leq 27$ [13] and $t(50) \leq 187$.

(5) In order to obtain $R(C_4, C_4, K_3, K_4) \leq 76$, proceed similarly using $R(C_4, C_4, K_3, K_3) \leq 36$ [13], $R(C_4, C_4, K_4) \leq 22$, and $t(76) \leq 348$.

(6) In order to obtain $R(C_4, C_4, K_4, K_4) \leq 179$, proceed similarly using $R(C_4, C_4, K_3, K_4) \leq 76$ and $t(179) \leq 1239$.

□

References

[1] N. Alon and V. Rödl, Sharp bounds for some multicolor Ramsey numbers, *Combinatorica,* 2005 (25) 125-141.

[2] D. Bevan, *personal communication* to S. Radziszowski, 2002.

[3] A. Bialostocki and J. Schönheim, On some Turán and Ramsey numbers for C_4, in *Graph Theory and Combinatorics* (ed. B. Bollobás), Academic Press, London 1984, 29-33.

[4] B. Bollobás, *Extremal Graph Theory.* Academic Press, London, 1978.

[5] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications.* American Elsevier Publishing Co., New York, 1976.

[6] C. R. J. Clapham, A. Flockhart and J. Sheehan, Graphs without four-cycles, *Journal of Graph Theory,* 1989 (13) 29-47.

[7] G. Exoo and D.F. Reynolds, Ramsey numbers based on C_5-decompositions, *Discrete Mathematics,* 1988 (71) 119-127.

[8] R. Irving, Generalised Ramsey numbers for small graphs, *Discrete Mathematics,* 1974 (9) 251-264.

[9] S. Radziszowski, Small Ramsey numbers. *Electronic Journal of Combinatorics*, Dynamic Survey 1, revision #11, August 2006, `http://www.combinatorics.org`.

[10] S. Radziszowski and Kung-Kuen Tse, A computational approach for the Ramsey numbers $R(C_4, K_n)$, *Journal of Combinatorial Mathematics and Combinatorial Computing*, 2002 (42) 195-207.

[11] C. -U. Schulte, Ramsey-Zahlen für Bäume und Kreise, *Ph.D. thesis*, Heinrich-Heine-Universität Düsseldorf (1992).

[12] Sun Yongqi, Yang Yuansheng, Lin Xiaohui and Zheng Wenping, The value of the Ramsey number $R_4(C_4)$, *Utilitas Mathematica,* 2007 (73) 33-44.

[13] X. Xu and S. Radziszowski, $28 \leq R(C_3, C_3, C_4, C_4) \leq 36$, *Utilitas Mathematica*. (to appear)

[14] Yang Yuansheng and P. Rowlinson, On extremal graphs without four-cycles, *Utilitas Mathematica*, 1992 (41) 204-210.