Graph Reconstruction Numbers

Brian McMullen and Stanisław P. Radziszowski
Department of Computer Science
Rochester Institute of Technology
Rochester, NY 14623
{bmm3056,spr}@cs.rit.edu

December 30, 2005

Abstract

Proposed in 1942, the Graph Reconstruction Conjecture posits that every simple, finite, undirected graph with three or more vertices can be reconstructed up to isomorphism to the original graph, given the multiset of subgraphs produced by deleting each vertex along with its incident edges. Related to this Reconstruction Conjecture, existential reconstruction numbers, $\exists rn(G)$, concern the minimum number of vertex-deleted subgraphs required to identify a graph up to isomorphism.

We discuss the resulting data from calculating reconstruction numbers for all simple, undirected graphs with up to ten vertices. From this data, we establish the reasons behind all high existential reconstruction numbers ($\exists rn(G) > 3$) for $|V(G)| \leq 10$ and identify new classes of graphs that have high reconstruction numbers for $|V(G)| > 10$.

We also consider 2-reconstructibility – the ability to reconstruct a graph G from the multiset of subgraphs produced by deleting each combination of two vertices from G. The 2-reconstructibility of all graphs with nine or less vertices was tested, identifying four graphs in this range with five vertices as the highest order of graphs that are not 2-reconstructible.

1 Introduction

In this paper, all graphs are assumed to be simple, finite and undirected. To distinguish between sets and multisets, $[., ..., .]$ denotes a multiset. Given graph G, $V(G)$ is the set of vertices of G and $|V(G)|$ is the order of G. If v is a vertex of G, then $G - v$ is the graph obtained from G by deleting vertex v and its incident edges – a vertex-deleted subgraph of G. The *deck* of G, $D(G)$, is the multiset of vertex-deleted subgraphs of G defined by $[G - v_0, ..., G - v_{n-1}]$ where $\{v_0, ..., v_{n-1}\} = V(G)$ and $n = |V(G)|$. Each member of $D(G)$ is referred to as a *card*. A *subdeck* of graph G is a multisubset of $D(G)$.

Proposed in 1942 by Kelly and Ulam, the Graph Reconstruction Conjecture states that for all graphs G and H with three or more vertices, $D(G) = D(H)$ *iff* G and H are isomorphic, denoted $G \cong H$ [4, 9]. Although open in the general case, the conjecture has been proven to hold for several classes of graphs such as disconnected graphs, regular graphs and trees [4, 8, 14]. Also, McKay confirmed by calculation that every graph G where $|V(G)| \leq 11$ is reconstructible, even when $D(G)$ is reduced by isomorphism type [10].

An issue related to the Reconstruction Conjecture is that of *reconstruction numbers*. While the Reconstruction Conjecture is concerned with the possibility of reconstructing any given graph up to isomorphism from its full deck, reconstruction numbers consider the subdeck size of a graph required to reconstruct it up to isomorphism. Specifically, in this paper we consider *existential reconstruction numbers*.

Definition 1. The *existential reconstruction number* of G, $\exists rn(G)$, is the minimum number of vertex-deleted subgraphs of G required to reconstruct G up to isomorphism.

It is known that in most cases, a given graph G is reconstructible from a small subset of cards from $D(G)$. In fact, Bollobás proved probabilistically that almost all graphs can be reconstructed with only three cards of a deck [3]. In this paper, for any graph G where $\exists rn(G) > 3$, G is considered to have a "high" reconstruction number. These appear to be relatively rare for small orders as well, as among more than twelve million graphs up to ten vertices there are only thirty-five of them.

It is also noteworthy that for any graph G on at least three vertices $\exists rn(G) \geq 3$, and in all cases $\exists rn(G) = \exists rn(\overline{G})$. In addition, several predictions can be made regarding the existential reconstruction numbers of disconnected graphs by the work of Myrvold and others [12, 13].

Theorem 1 ([12, 13]). *Given disconnected graph G, where not all connected components are isomorphic, $\exists rn(G) = 3$.*

Theorem 2 ([13]). *Given disconnected graph $G = pK_c$, $\exists rn(G) = c + 2$, for $p, c \geq 2$.*

In other words, graphs of the form pH are the only disconnected graphs where high $\exists rn$ values are possible. Among these, for any graph $G = pK_c$ with $p, c \geq 2$, $\exists rn(G) = c + 2$. In 2002, Asciak and Lauri proved that this is the only class of disconnected graphs with such high $\exists rn$ values relative to $|V(H)|$.

Theorem 3 ([1]). *Given disconnected graph $G = pH$, where H is not a complete subgraph, $\exists rn(G) \leq |V(H)|$.*

We calculated existential reconstruction numbers for all graphs with up to ten vertices. This was achieved by our algorithms using the *nauty* software package developed by McKay [11] and *Condor* created at the Computer Science Department at the University of Wisconsin [5]. *Nauty* provided efficient isomorph-free exhaustive generation of graphs and identification of isomorphs. *Condor* was used to perform calculations in parallel across machines on the RIT CS department local network for efficiency. A sketch of the algorithm used to find reconstruction numbers is given in Section 3.

2 Results

2.1 Existential Reconstruction Number Counts

Table 1 shows the counts of graphs with their determined $\exists rn$ values arranged by order n.

Table 1: $\exists rn(G)$ counts

$\exists rn$ \ n	3	4	5	6	7	8	9	10
3	4	8	34	150	1044	12,334	274,666	12,005,156
4		3		4		8		6
5				2		2	2	4
6						2		
7								2

In addition to finding $\exists rn$ value counts, the software also recorded every graph G with $\exists rn(G) > 3$. By analyzing these graphs, we were able to determine the reason behind the high existential reconstruction number found for every graph G with $|V(G)| \leq 10$. With the exception of one counter-example, every graph found with a high $\exists rn$ value could be placed in one of a few classes. Most of these classes can be used to identify an infinite number of graphs of larger order with high existential reconstruction numbers. Determining upper bounds on $\exists rn(G)$ is usually even harder, and is not addressed in the proofs of this paper. The exact values of $\exists rn(G)$ listed here are the results of exhaustive computations.

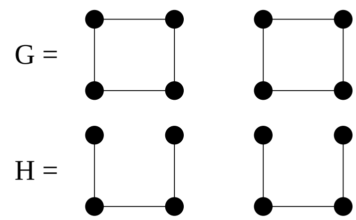

Figure 1: $\exists rn(G) = \exists rn(H) = 4$

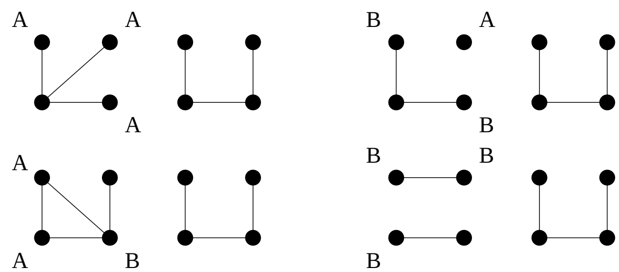

Figure 2: Four graphs proving $\exists rn(H) > 3$

2.2 Classes of Graphs with a High $\exists rn$

2.2.1 Disconnected Graphs of the Form pK_c

It is already known from Theorem 2 that there exist graphs G in this class such that $\exists rn(G) > 3$ since $\exists rn(pK_c) = c + 2$. Graphs $2K_2$, $2K_4$, $4K_2$, $2K_3$, $3K_2$, $3K_3$, $2K_5$, $5K_2$ were all calculated to have their expected reconstruction numbers. And since $\exists rn(G) = \exists rn(\overline{G})$ for every graph G, their complements were identified as having high reconstruction numbers as well.

2.2.2 Other Disconnected Graphs of the Form pH

It is not true that for every graph $G = pH$, where H is not complete, $\exists rn(G) > 3$, however only two such cases were found from our calculations. These are given in Figure 1.

Given what we already know regarding reconstruction numbers, the $\exists rn$ value of graph $2C_4$ (G from Figure 1), is not surprising. $C_4 \cong K_{2,2} \cong \overline{2K_2}$, and $D(K_1 \cup K_3)$ has three cards in common with $D(2K_2)$. As one might

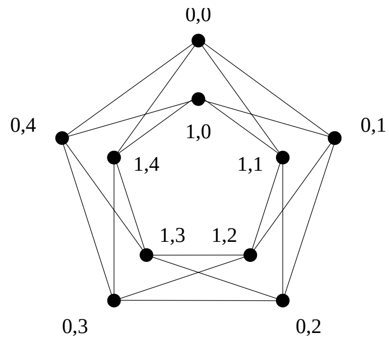

Figure 3: $\exists rn(G) = 4$

expect, $D(\overline{K_1 \cup K_3})$ has three cards in common with C_4. It is because $D(2K_2)$ and $D(C_4)$ are both multisets of isomorphic cards that we require only one graph with three cards from its deck in common with theirs to prove that they have high existential reconstruction numbers.

It is also true that $D(2C_4)$ is a multiset of isomorphic cards. So by constructing graph $\overline{K_1 \cup K_3} \cup C_4$, we have a graph with three cards in its deck isomorphic to $2C_4$, proving that $\exists rn(2C_4) > 3$. By Theorem 3, it must be true that $\exists rn(2C_4) \leq 4$ as well.

However, the high $\exists rn$ value for graph H does not seem to be so easy to explain. We use several graphs to determine that all possible multisubsets of $D(H)$ with three elements are also shared by the decks of other graphs. Such graphs are given in Figure 2.

Deleting each vertex from the top row of H in Figure 1 creates a set of four isomorphic cards which we label A, and deleting vertices from the bottom row of H leaves us with another set of isomorphs to be labeled B. Figure 2 shows four graphs nonisomorphic to H whose decks contain all possible multisubsets with three elements from $D(H)$. The labeled vertices indicate those which can be deleted to recreate cards A or B from $D(H)$.

2.2.3 Redundantly Connected Cycles

Redundantly connected cycles in this paper refer to a new class of graphs identified as having high existential reconstruction numbers. Notation $v_a \sim v_b$ states that vertex a is connected to vertex b and, in the following definition, $v_{c,i}$ is the i^{th} vertex of cycle c.

Definition 2. We define $RCC_{n,j}$ to be a graph with $n \geq 2$ "redundantly connected cycles", each of length $j \geq 3$. To compose graph $RCC_{n,j}$ we

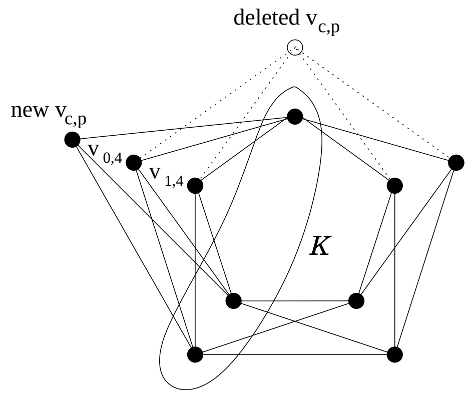

Figure 4: Graph H for $RCC_{2,5}$

begin with graph $F = nC_j$. In each component of F, $v_{c,i} \sim v_{c,(i+1)\ mod\ j}$ where $0 \leq i \leq j-1$ and $0 \leq c \leq n-1$. Then by adding edges between the cycles of F such that $v_{c,i} \sim v_{d,(i+1)\ mod\ j}$ for $c \neq d$, we create graph $RCC_{n,j}$.

G in Figure 3 is the graph $RCC_{2,5}$. This was a sample RCC graph identified by computations with a high $\exists rn$ value. However, given the following theorem, we see that for every graph H in this class $\exists rn(H) > 3$.

Theorem 4. *$\exists rn(RCC_{n,j}) > n+1$, for all $n \geq 2$ and $j \geq 3$.*

Proof. For any graph $G = RCC_{n,j}$, $D(G)$ is a multiset of isomorphic subgraphs. Therefore, to prove $\exists rn(G) > n+1$ we need only construct one graph $H \ncong G$ such that $D(H)$ shares $n+1$ elements with $D(G)$

To construct H, first delete any single vertex $v_{c,p}$ from G, giving us a card from $D(G)$. Next, choose any set of vertices from the resulting subgraph labeled $v_{d,q}$, where $d < n$, and q is fixed, $q \neq p$. All vertices from this set are connected to the same set of vertices $\mathcal{K}$. By re-adding deleted vertex $v_{c,p}$ and placing edges connecting it to each $v \in \mathcal{K}$, we create graph H.

It is easy to see that $H - v_{c,p}$ is isomorphic to a card from $D(G)$. However, because $v_{c,p}$ is connected to every vertex in $\mathcal{K}$, $H - v_{d,q}$ is also isomorphic to a card from $D(G)$ for each d. Also, since H is not regular, $H \ncong G$. $\square$

As an example, Figure 4 shows graph H where $D(H)$ shares three cards in common with $D(RCC_{2,5})$, obtained by deleting vertices $v_{0,4}$, $v_{1,4}$ and *new* $v_{c,p}$.

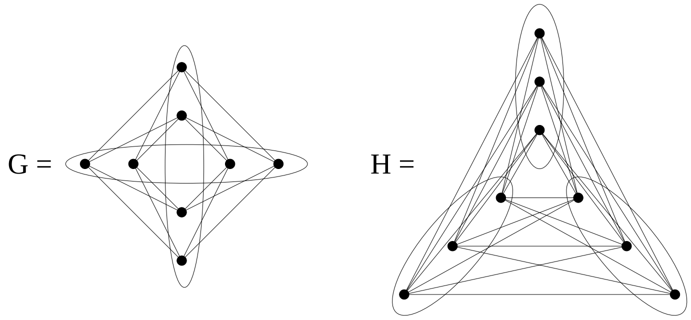

Figure 5: $G = RCC_{2,4} = K_{4,4}$ and $H = RCC_{3,3} = K_{3,3,3}$

$RCC_{2,5}$ was the first and only graph in our results which led to the RCC class. The astute reader may be curious why this was the case as $RCC_{2,3}$, $RCC_{2,4}$ and $RCC_{3,3}$ are in the range of graphs with ten or less vertices as well. The reason for this is that they are also complements of pK_c, and their high $\exists rn$ values were already explained. Overlaps between RCC graphs and pK_c complements are discussed below.

Remark 1. $RCC_{n,4} \cong K_{2n,2n}$

Proof. All vertices of graph $RCC_{n,4}$ labeled $v_{c,p}$ where p is even are connected to all vertices $v_{c,q}$, where q is odd, and no others. □

Remark 2. $RCC_{n,3} \cong K_{n,n,n}$

Proof. All vertices of $RCC_{n,3}$ labeled $v_{c,p}$ are connected to all vertices $v_{c,q}$, where $p \neq q$, and no others. □

These proofs are illustrated in Figure 5 with bipartite and tripartite groupings of vertices indicated. At this point we have no clear idea on how to establish in general the exact value of $\exists rn(RCC_{n,j})$ for all n, j.

2.2.4 Pairs of Complete Graphs Connected by bK_2

To discuss this class of graphs, a new notation is helpful.

Definition 3. For $c_1, c_2 \geq 1$, $1 \leq b \leq min(c_1, c_2)$, graph $K_{c_1} \leftrightarrow^b K_{c_2}$ is formed by disjoint graphs K_{c_1} and K_{c_2} with b additional edges forming partial matching between vertices of the two complete subgraphs.

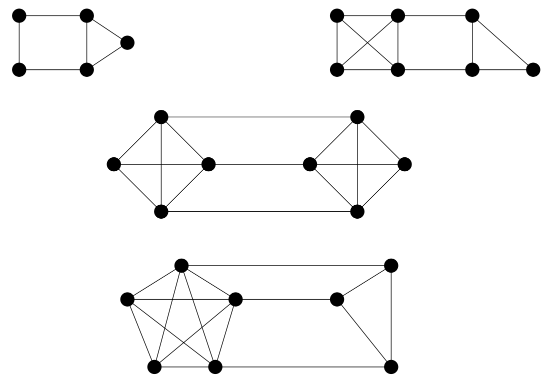

Figure 6: Examples of $K_{c_1} \leftrightarrow^b K_{c_2}$ graphs

Because the two subgraphs being connected are complete, the orders of the subgraphs and the number of one-to-one edges connecting them specifies a unique graph up to isomorphism. The graphs $K_2 \leftrightarrow^2 K_3$, $K_4 \leftrightarrow^2 K_3$, $K_4 \leftrightarrow^3 K_4$ and $K_5 \leftrightarrow^3 K_3$ are presented in Figure 6.

Theorem 5 below specifies a range of values of c_1, c_2 and b for $G = K_{c_1} \leftrightarrow^b K_{c_2}$, for which it must be the case that $\exists rn(G) > 3$.

Theorem 5. *For any graph $G = K_c \leftrightarrow^b K_c$, where $c \geq 3$ and $2 \leq b \leq c-1$, $\exists rn(G) > 3$.*

Proof. Given graph $G = K_c \leftrightarrow^b K_c$, $D(G)$ contains $2b \geq 4$ copies of $K_c \leftrightarrow^{b-1} K_{c-1}$, subgraph A, and $2(c-b) \geq 2$ copies of $K_c \leftrightarrow^b K_{c-1}$, subgraph B, exclusively. Therefore, to prove $\exists rn(G) > 3$ we must show that there exist graphs nonisomorphic to G whose decks contain three elements of A, two of A and one of B, two of B and one of A, and finally three copies of B, whenever such combinations are possible in $D(G)$.

Given graph $H = K_{c+1} \leftrightarrow^{b-1} K_{c-1}$, a subgraph isomorphic to A is created for each vertex deleted from K_{c+1} not connected to K_{c-1}. This means that H contains $c - b + 2 \geq 3$ copies of A in its deck.

With graph $I = K_{c+1} \leftrightarrow^b K_{c-1}$, we obtain $b \geq 2$ copies of A and $c - b + 1 \geq 2$ copies of B by deleting each vertex from K_{c+1}. Therefore, from $D(I)$ we can extract a multisubset with two copies of A and one copy of B and a multisubset with two copies of B and one copy of A.

Finally we consider $J = K_{c+1} \leftrightarrow^{b+1} K_{c-1}$. By deleting vertices from K_{c+1} we can obtain $b+1 \geq 3$ copies of B. (Note that $K_{c+1} \leftrightarrow^{b+1} K_{c-1}$ has parameters violating Definition 3 for $b = c-1$. But since there are only two copies of B in the multiset $D(K_c \leftrightarrow^b K_c)$ for $b = c-1$, the construction of J is not necessary in this case.)

Table 2: $\exists rn$ values for $K_c \leftrightarrow^b K_c$, $3 \leq c \leq 5$

G	$\exists rn(G)$
$K_3 \leftrightarrow^2 K_3$	4
$K_4 \leftrightarrow^2 K_4$	4
$K_4 \leftrightarrow^3 K_4$	5
$K_5 \leftrightarrow^2 K_5$	4
$K_5 \leftrightarrow^3 K_5$	5
$K_5 \leftrightarrow^4 K_5$	5

The constructions of graphs H, I and J imply $\exists rn(G) > 3$, since all of them are clearly not isomorphic to $K_c \leftrightarrow^b K_c$. $\square$

The calculated $\exists rn(G)$ values for all graphs in this class on up to ten vertices are given in Table 2. At this point we have no simple method for determining the exact value of $\exists rn(K_c \leftrightarrow^b K_c)$ based on b and c.

2.3 High $\exists rn$ Exception

The only graph with a high existential reconstruction number that did not fit in the above categories is P_4 with $\exists rn(P_4) = 4$. This can easily be proven as follows. There are two possible multisubsets of order three in $D(P_4)$. The two graphs that share each of these subdecks in their own decks are shown in Figure 7. Since P_4 has so few vertices, it may just be a degenerate case. Note also that $P_4 \cong \overline{P_4}$, which explains the odd number of graphs G of order four with $\exists rn(G) = 4$.

3 Algorithm

The following is an enhanced form of an algorithm used by Baldwin for her own Master's Project on graph reconstruction numbers [2].

The algorithm for calculating $\exists rn(G)$ for graph G involves considering all graphs whose decks have at least three cards in common with $D(G)$. These are found by first taking each unique subgraph H from $D(G)$ and determining every graph I, where $I - v \cong H$ for some vertex v. Since this technique is bound to generate isomorphs, a canonical labeling of each new I that is calculated is placed in a hash table.

As each new graph I is added to the hash table, $D(I)$ is determined. Each card from $D(I)$ that is found to be isomorphic to a card from $D(G)$ that has not yet been matched up is marked. If three or more cards from $D(I)$ are matched with cards from $D(G)$ the results are added as a row to

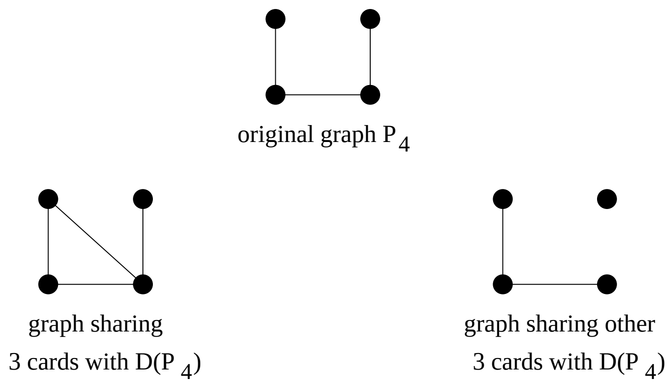

Figure 7: $\exists rn(P_4) = 4$

a *relation matrix*. When completed, the rows from the relation matrix will represent the matches between the decks of all graphs I and $D(G)$, where $D(I)$ shares at least three cards with $D(G)$.

With the relation matrix completed, we use it to test each possible subdeck of $D(G)$ of size n, beginning with $n = 3$, to see if it is unique to $D(G)$. If we find one subdeck that is not in common with any $D(I)$ in the relation matrix, then $\exists rn(G) = n$. If all subdecks of size n are contained in the deck of at least one of the I graphs, then n is incremented and testing begins for all possible combinations of the new subdeck size. As long as G is reconstructible, a value for $\exists rn(G)$ will be found before $n > |V(G)|$.

When implementing the algorithm, computations were performed individually for each group of graphs of a given order and fixed number of edges. By doing this, we were able to take advantage of the fact that $\exists rn(G) = \exists rn(\overline{G})$ by computing $\exists rn$ for all graphs below the appropriate edge count and simply assigning that value to their complements. In this way the time needed to compute existential reconstruction numbers was almost halved.

4 2-Reconstructibility

Kelly proposed that the Reconstruction Conjecture can be generalized by considering the reconstructibility of graphs from subgraphs created by deleting some number k vertices instead of just one [7, 8]. The deck of graph G where each card is created by deleting a combination of k vertices is denoted as $D_k(G)$. Graph G is said to be *k-reconstructible iff* it is reconstructible up to isomorphism given $D_k(G)$.

In addition to finding existential reconstruction numbers, calculations

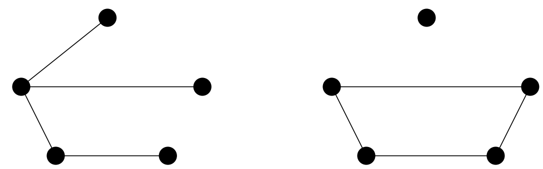

Figure 8: Two graphs with the same 2-vertex deleted decks

testing 2-reconstructibility were completed for all graphs G up to nine vertices. In these calculations, seven of the eleven possible graphs with four vertices were found not to be 2-reconstructible. This is not surprising considering the low number of vertices from the original graph compared to the number being deleted.

There were four graphs found with five vertices that were not 2-reconstructible. These were the two graphs pictured in Figure 8 and their complements. It is easy to verify by hand that these two graphs have the same 2-vertex deleted decks.

For all graphs G, where $6 \leq |V(G)| \leq 9$, G was found to be 2-reconstructible. So as the Reconstruction Conjecture essentially poses that for every graph G, where $|V(G)| \geq 3$, G is 1-reconstructible, we might also wonder whether for every graph H, where $|V(H)| \geq 6$, H is 2-reconstructible.

5 Further Work

The resulting data and analysis from this paper point to several topics that are worth further research. Most interesting of these is whether there exists a graph G of odd prime order such that $\exists rn(G) > 3$, a question that has been brought up elsewhere [6]. In the same paper it has also been suggested that there are no graphs G with an odd number of vertices where $\exists rn(G) > 3$, but this is not the case as implied by Theorem 2. The question of prime order graphs becomes even more interesting with our results, since every class of graphs found here having high existential reconstruction numbers requires that its members have a composite number of vertices. Such graphs of prime order are very likely to exist, yet finding one with the smallest order seems to be not easy at all.

Other considerations could include what other classes of graphs might exist whose members must have high $\exists rn$ values. Also, we might try to think of a simple method for precisely determining $\exists rn(K_c \leftrightarrow^b K_c)$ and $\exists rn(RCC_{n,j})$ in general. Finally, as asked in Section 4, does there exist any graph G with $|V(G)| \geq 6$ such that G is not 2-reconstructible?

References

[1] K. J. Asciak and J. Lauri, On Disconnected Graph with Large Reconstruction Number. *Ars Combinatoria*, 62:173-181, 2002.

[2] J. Baldwin, *Graph Reconstruction Numbers*, Master's Project, Department of Computer Science, Rochester Institute of Technology, 2004.

[3] B. Bollobás, Almost every graph has reconstruction number three. *Journal of Graph Theory*, 14(1):1-4, 1990.

[4] J. A. Bondy, A Graph Reconstructor's Manual. *Surveys in Combinatorics*, 1991 (Guildford, 1991), 221-252, London Math. Soc. Lecture Note Ser., 166, Cambridge Univ. Press, Cambridge, 1991.

[5] Condor Team, University of Wisconsin-Madison, *Condor Version 6.6.7 Manual*, May 2004.

[6] F. Harary and M. Plantholt, The Graph Reconstruction Number. *Journal of Graph Theory*, 9:451-454, 1985.

[7] E. Hemaspaandra, L. Hemaspaandra, S. Radziszowski and R.Tripathi, Complexity Results in Graph Reconstruction. *Proceedings of the 29th International Symposium on Mathematical Foundations of Computer Science*. To appear.

[8] P. J. Kelly, A congruence theorem for trees. *Pacific Journal of Mathematics*, 7:961-968, 1957.

[9] P. J. Kelly, *On Isometric Transformations*. Ph.D. thesis, University of Wisconsin, 1942.

[10] B. D. McKay, Small graphs are reconstructible. *Australs. Journal of Combinatorics*, 15:123-126, 1997.

[11] B. D. McKay, *Nauty User's Guide*, Version 2.2 (1984-2004). Computer Science Department, Australian National University, `http://cs.anu.edu.au/~bdm/nauty/`.

[12] R. Molina, Correction of a proof on the ally-reconstruction number of disconnected graphs. *Ars Combinatoria*, 20:59-64, 1995.

[13] W. J. Myrvold, The ally reconstruction number of a disconnected graph. *Ars Combinatoria*, 28:123-127, 1989.

[14] C. St. J. A. Nash-Williams, The Reconstruction Problem. *Selected Topics in Graphs Theory*, Academic Press, San Diego, 205-236, 1978.