# Computation of the Ramsey Number $R(B_3, K_5)$

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**Abstract**

In 1989, George R. T. Hendry presented a table of two-color graph Ramsey numbers $R(G,H)$ for all pairs of graphs $G$ and $H$ having five vertices, with the exception of seven cases. Until now, only two of these open cases were solved. This work eliminates another one by computing $R(B_3,K_5) = 20$, where $B_3 = K_2 + \overline{K}_3$ is the book graph of order 5. In addition, we show that for these parameters there exists a unique up to isomorphism critical graph. The results are based on computer algorithms. Among the four remaining open cases in Hendry's table, the most notable is that of $K_5$ versus $K_5$, for which it is known that $43 \le R(K_5,K_5) \le 49$.

**Keywords:** Ramsey numbers, graph algorithms
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## 1. Overview

Let $G$ and $H$ be simple graphs. Graph $F$ will be called a $(G,H)$-graph if it does not contain a subgraph isomorphic to $G$, and such that the complement $\overline{F}$ has no subgraph isomorphic to $H$. A $(G,H;n)$-graph is a $(G,H)$-graph of order $n$. Let $\mathcal{R}(G,H)$ and $\mathcal{R}(G,H;n)$ denote the set of all $(G,H)$-graphs and $(G,H;n)$-graphs, respectively. The Ramsey number $R(G,H)$ is defined to be the least $n > 0$ such that there is no $(G,H;n)$-graph, or equivalently, it is the least positive integer $n$ such that every

2-coloring of the edges of $K_n$ contains a subgraph isomorphic to $G$ in the first color or a subgraph isomorphic to $H$ in the second color.

In the sequel, $P_k$ and $C_k$ denote a simple path and cycle, respectively, on $k$ vertices. Further, $B_k = K_2 + \overline{K}_k$ is the $k$-page book of order $k+2$, which can be seen as a graph formed by $k$ triangular pages sharing one common edge. Note that we could define equivalently $B_k = K_1 + K_{1,k}$, where $K_{1,k}$ is a star with $k$ spikes. Finally, let $W_k = K_1 + C_{k-1}$ denote a wheel with $k-1$ spokes.

In 1989, George R. T. Hendry [1] presented a table of Ramsey numbers $R(G,H)$ for all pairs of graphs $G$ and $H$ having five vertices, with the exception of seven cases. Until now, only two of these open cases have been solved. A regularly updated survey by the second author [4] reports on old and the most recent results on various types of Ramsey numbers, including those of the form $R(G,H)$. In particular, [4] lists the developments related to all seven cases missing in the Hendry's table, and gives references to papers discussing them. For graphs $G$ and $H$ of order less than 5, or $G$ of order at most 4 and $H$ of order 5, all Ramsey numbers $R(G,H)$ have been known since the computation of $R(K_4,K_5)=25$ [3] in 1995.

In this work, we eliminate one of the open cases by computing $R(B_3,K_5)=20$. This result improves the bounds $20 \le R(B_3,K_5) \le 22$ given in [1]. In addition, we show that for these parameters there exists a unique up to isomorphism critical graph, i.e. $|\mathcal{R}(B_3,K_5;19)|=1$. We have also completed a full enumeration of all $(B_3,K_5;18)$-graphs, and found that there are exactly 3376 of them. Our results, described in more detail in the next section, are based on computer algorithms.

The remaining open cases of Ramsey numbers for graphs on at most five vertices are: $25 \le R(K_5-P_3,K_5) \le 28$, $27 \le R(W_5,K_5) \le 29$, $30 \le R(K_5-e,K_5) \le 34$, and $43 \le R(K_5,K_5) \le 49$ (see [4] for references to all bounds). Among them, definitely the most famous one and the most difficult to compute, is that of $K_5$ versus $K_5$. While some very strong evidence has been presented to support the conjecture that $R(K_5,K_5) = 43$, there is little hope for a computational proof of the latter, or even for any improvement of the current lower or upper bound. In contrast, further progress on improving bounds for the other three open cases can be expected in the not so distant future. More specifically, we speculate as follows. We feel that the case of $R(W_5,K_5)$ could be attacked with a method similar to the approach in this paper, though more computational effort would be needed. The evaluation of $R(K_5-P_3,K_5)$ might be at least as hard to obtain as the result $R(K_4,K_5)=25$, since $K_5-P_3$ contains $K_4$.

The computation of the exact value of $R(K_5 - e, K_5)$ seems to be still much more difficult, but some improvement of the current bounds (especially upper) should be possible.

## 2. Enumerations and Results

We will use the same notation as in [5]. If $G$ is a graph, then $VG$ and $EG$ are its vertex set and edge set, respectively. If $v \in VG$, then $N_G(v) = \{w \in VG \mid vw \in EG\}$, and $\deg_G(v) = |N_G(v)|$. The subgraph of $G$ induced by $W$ will be denoted by $G[W]$. Also, for $v \in VG$, define the induced subgraphs $G_v^+ = G[N_G(v)]$ and $G_v^- = G[VG - N_G(v) - \{v\}]$.

Note that if $G \in \mathcal{R}(B_3, K_m; n)$ and $v \in VG$, then necessarily $G_v^+ \in \mathcal{R}(K_{1,3}, K_m; d)$, where $d = \deg_G(v)$, and $G_v^- \in \mathcal{R}(B_3, K_{m-1}; n-d-1)$. Hence, $G_v^+$ must be simply a disjoint union of paths and cycles, and $G_v^-$ is of the same type as $G$, but for $m-1$.

It is a straightforward exercise to find all 104 graphs in $\mathcal{R}(K_{1,3}, K_5)$, and fairly simple algorithms are sufficient to generate all 65694 graphs in $\mathcal{R}(B_3, K_4)$. The statistics of both families by the number of graphs with fixed number of vertices is given in Table I. Observe that $R(K_{1,3}, K_5) = 13$ and $R(B_3, K_4) = 14$.

| $s$ | $\lvert\mathcal{R}(K_{1,3}, K_5; s)\rvert$ | $\lvert\mathcal{R}(B_3, K_4; s)\rvert$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 2 | 2 |
| 3 | 4 | 4 |
| 4 | 7 | 10 |
| 5 | 10 | 25 |
| 6 | 16 | 86 |
| 7 | 19 | 326 |
| 8 | 21 | 1518 |
| 9 | 14 | 7000 |
| 10 | 7 | 23462 |
| 11 | 2 | 28838 |
| 12 | 1 | 4410 |
| 13 | | 12 |
| total | 104 | 65694 |

**Table I.** Statistics for $\mathcal{R}(K_{1,3}, K_5)$ and $\mathcal{R}(B_3, K_4)$.

Using the graphs reported in Table I as $G_v^+$ and $G_v^-$, the family of graphs $\mathcal{R}(B_3, K_5; n)$ can be constructed by applying a gluing algorithm to $G_v^+ \in \mathcal{R}(K_{1,3}, K_5; s)$ and $G_v^- \in \mathcal{R}(B_3, K_4; t)$ for all possible $s$ and $t$ satisfying $s+t+1=n$. The gluing algorithm used in this work was simpler than, but similar to, that described in [3, 5]. Also, it had to include some modifications needed in order to avoid the graph $B_3$ instead of $K_4$ or $C_4$. The computations were completed in three stages as follows.

All $(B_3, K_5; 18)$-graphs were obtained by performing gluing of graphs $G_v^+$ to $G_v^-$ as above for $s \in \{4,5,6,7,8\}$ and $t = 17-s$. No gluing for $s \geq 9$ needs to be done, since it is easy to see that no $(B_3, K_5; 18)$-graph can have minimum degree 9 or higher. The statistics of results by the number of edges and the minimum degree is presented in Table II.

All $(B_3, K_5; 19)$-graphs were obtained in two ways: by performing gluing as above for $s \in \{5,6,7,8\}$, $t = 18-s$, and independently by constructing and $(B_3, K_5)$-filtering one-vertex extensions of all 3376 $(B_3, K_5; 18)$-graphs. Both paths led to the same unique $(B_3, K_5; 19)$-graph, which is cyclic and regular of degree 6, with the edges connecting pairs of vertices belonging to $\mathcal{Z}_{19}$ in distances 1, 7, and 8.

| $d$ / $e$ | 4 | 5 | 6 | 7 | total |
|---|---|---|---|---|---|
| 51 | | 2 | | | 2 |
| 52 | | 6 | | | 6 |
| 53 | | 42 | | | 42 |
| 54 | 1 | 195 | 19 | | 215 |
| 55 | 14 | 448 | 65 | | 527 |
| 56 | 32 | 572 | 208 | | 812 |
| 57 | 19 | 439 | 321 | | 779 |
| 58 | 2 | 246 | 252 | | 500 |
| 59 | | 121 | 142 | | 263 |
| 60 | | 60 | 82 | | 142 |
| 61 | | 19 | 40 | | 59 |
| 62 | | 3 | 16 | | 19 |
| 63 | | | 4 | 6 | 10 |
| total | 68 | 2153 | 1149 | 6 | 3376 |

**Table II.** Statistics for $(B_3, K_5; 18)$-graphs, $e = |E|$, $d$ is the minimum degree.

Similarly, all $(B_3, K_5; 20)$-graphs were obtained in two ways: by performing gluing as above for $s \in \{6,7,8,9\}$, $t = 19-s$, and independently by constructing and $(B_3, K_5)$-filtering all one-vertex extensions of the unique $(B_3, K_5; 19)$-graph. Both paths led to no graphs, and thus $R(B_3, K_5) = 20$.

The graphs $\mathcal{R}(B_3, K_5; 18)$ are really not needed, neither is the uniqueness of the critical graph, to claim that $R(B_3, K_5) = 20$. However, these provided additional strong correctness tests of the enumeration results, since several graphs and graph families were generated more than once by different algorithms.

**Theorem.** $R(B_3, K_5) = 20$.

**Proof.** The computations and results described above prove that there does not exist any $(B_3, K_5; 20)$-graph, so $R(B_3, K_5) \leq 20$. It is easy to verify that a cyclic graph with the edges joining vertices belonging to $\mathcal{Z}_{19}$, which are in distance 1, 7 or 8, has no $B_3$ and no $\overline{K}_5$. This implies the lower bound. ∎

Three separate implementations of the algorithms were prepared and their results compared. The computations were performed as a part of the MS project by the first author, were verified by the second author, and, independently, all reported graph families were enumerated by the third author. The computational effort of this project was moderate — all computations could now be repeated overnight on a local departmental network (150+ machines). If we were only computing the value of $R(B_3, K_5)$, without completing the exhaustive enumeration of all $(B_3, K_5; \geq 18)$-graphs, then the whole computation would be still much faster.

A general utility program for graph isomorph rejection, *nauty* [2], together with other graph manipulation tools, written by Brendan McKay, was used extensively. All graphs in $\mathcal{R}(B_3, K_5; 18)$ are available from the second author.

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