Computation of the Folkman Number $F_e(3, 3; 5)$

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Abstract. With the help of computer algorithms, we improve the lower bound on the edge Folkman number $F_e(3, 3; 5)$ and vertex Folkman number $F_v(3, 3; 4)$, and thus show that the exact values of these numbers are 15 and 14, respectively. We also present computer enumeration of all critical graphs.

1 Introduction

Let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. Let r and l be positive integers. We write $G \rightarrow (r, l)^v$ ($G \rightarrow (r, l)^e$) if every red-blue coloring of the vertices (edges) of G forces a red complete subgraph K_r or a blue complete subgraph K_l in G. For $p > \max\{r, l\}$, let

$$\mathcal{F}_v(r, l; p) = \{G : G \rightarrow (r, l)^v \wedge K_p \nsubseteq G\}$$

and

$$\mathcal{F}_e(r, l; p) = \{G : G \rightarrow (r, l)^e \wedge K_p \nsubseteq G\}.$$

The graphs in $\mathcal{F}_v(r, l; p)$ are called *vertex Folkman graphs*, and the graphs in $\mathcal{F}_e(r, l; p)$ are called *edge Folkman graphs*.

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It is well known that $K_6 \rightarrow (3,3)^e$, and so $K_6 \in \mathcal{F}_e(3,3;p)$ for all $p > 6$. In 1967 Erdős and Hajnal [2] asked if $\mathcal{F}_e(3,3;6) \neq \emptyset$, and the next year Graham [6] answered this question showing that $K_8 - C_5 \in \mathcal{F}_e(3,3;6)$, where $K_s - C_t$ is the graph obtained by deleting the edges of a cycle C_t from K_s. In 1970 Folkman [4] showed that for all r, l and $p > \max(r,l)$ the families $\mathcal{F}_v(r,l;p)$ and $\mathcal{F}_e(r,l;p)$ are nonempty. One can ask what is the minimum number of vertices in a vertex or edge Folkman graph, which leads to the notion of Folkman numbers. Let us denote

$$F_v(r,l;p) = \min\{|V(G)| : G \in \mathcal{F}_v(r,l;p)\}$$

and

$$F_e(r,l;p) = \min\{|V(G)| : G \in \mathcal{F}_e(r,l;p)\}.$$

These numbers are called *vertex Folkman numbers* and *edge Folkman numbers*, respectively. Observe that, for $p > r + l - 1$, we have $F_v(r,l;p) = r + l - 1$ as a trivial consequence of the pigeon-hole principle. More about vertex Folkman numbers can be found in [13] and [14]. Since the clique on $R(r,l)$ vertices has the smallest number of vertices among graphs G with the property $G \rightarrow (r,l)^e$, obviously we have $F_e(r,l;p) = R(r,l)$ for every $p > R(r,l)$ ($R(r,l)$ is the Ramsey number, i.e. the smallest integer n such that $K_n \rightarrow (r,l)^e$). Lin [12] proved that in some cases we have the equality $F_e(r,l;R(r,l)) = R(r,l) + 2$; in particular Lin's theorem implies $F_e(3,3;6) = 8$, $F_e(3,5;14) = 16$, and $F_e(4,4;18) = 20$. The only other previously known nontrivial exact result was obtained by Nenov [19] who proved $F_e(3,4;9) = 14$. Very little is known about the edge Folkman numbers in the case $p < R(r,l)$.

In this paper we compute the smallest unknown Folkman number $F_e(3,3;5)$. Table 1 summarizes the history of it's bounds. The first proof of the existence of this number is due to Pósa (unpublished). Schäuble [20] in 1969 showed that $F_e(3,3;5) \leq 42$. This upper bound was improved in 1971 by Graham and Spencer [7] to $F_e(3,3;5) \leq 23$, and they conjectured that $F_e(3,3;5) = 23$, but as they admitted, without much evidence. Their bound was pushed down to 18 by Irving [11] in 1973. In 1979 Hadziivanov and Nenov [8] constructed a 16-vertex graph in $\mathcal{F}_e(3,3;5)$, and in 1981 Nenov [18] presented the first 15-vertex graph with that property, proving that $F_e(3,3;5) \leq 15$. The second one was found in 1984 by Hadziivanov and Nenov [9]. The last three papers (written in Russian) had not been generally noticed at that time. In 1993 Erickson [3] found a 17-vertex graph in $\mathcal{F}_e(3,3;5)$ and conjectured that $F_e(3,3;5) = 17$. This was recently disproved by Bukor [1], who came up with the same 16-vertex graph as in [8]. In 1996 Urbański [22] showed another construction of the 15-vertex graph from [9].

Table 1. The History of Bounds on $F_e(3,3;5)$.

year	reference	lower	upper
1967	P. Erdős, A. Hajnal [2]		?
1969	M. Schäuble [20]		42
1971	R. L. Graham, J. H. Spencer [7]		23
1972	S. Lin [12]	10	
1973	R. W. Irving [11]		18
1979	N. Hadziivanov, N. Nenov [8]		16
1980	N. Nenov [17]	11	
1981	N. Nenov [18]		15
1985	N. Hadziivanov, N. Nenov [10]	12	
1993	M. Erickson [3]		17
1994	J. Bukor [1]		16
1998	this work	15	15

As far as the lower bound is concerned, in 1972 Lin [12] showed that $F_e(3,3;5) \geq 10$ and his result was later improved by Nenov [17] to $F_e(3,3;5) \geq 11$, and by Hadziivanov and Nenov [10] to $F_e(3,3;5) \geq 12$.

Much less is known about the number $F_e(3,3;4)$. Frankl and Rödl [5] proved that $F_e(3,3;4) \leq 10^{12}$ and later Spencer [21] squeezed out from their proof the inequality $F_e(3,3;4) \leq 10^{10}$. No reasonable lower bound for this Folkman number is known.

In the sequel we use the following notation. For an arbitrary graph G: $G+e$ denotes the graph with $V(G+e) = V(G)$ and $E(G+e) = E(G) \cup e$, for any edge e. $G[S]$ is the subgraph induced by the vertex set $S \subseteq V(G)$. $N_G(v)$ and $\deg_G(v)$ denote the neighborhood of vertex v and the degree of vertex v, respectively.

Let $\mathcal{F}_e(r,l;p;n) = \{G \in \mathcal{F}_e(r,l;p) : |V(G)| = n\}$. We say that G is an $(r,l;p;n)^e$ graph if and only if $G \in \mathcal{F}_e(r,l;p;n)$. Similary, we define the family $\mathcal{F}_v(r,l;p;n)$ of $(r,l;p;n)^v$ graphs. The graphs in these families with the smallest number of vertices, namely $F_v(r,l;p)$ and $F_e(r,l;p)$, respectively, are called *critical*.

2 The Algorithm

Our computational approach is based on the properties gathered in the following lemmas.

Lemma 1. $G \in \mathcal{F}_e(3,3;5) \Rightarrow \chi(G) \geq 6$

Proof. Assume that $V(G)$ can be partitioned into 5 independent sets $A_0, ..., A_4$. Color all the edges in $\{\{v,w\} : v \in A_i, w \in A_{(i+1) \bmod 5}\}$ with color 1 and all remaining edges with color 2. Since there are no monochromatic triangles we have $G \notin \mathcal{F}_e(3,3;5)$. ■

$R(5,3) = 14$ implies the following lemma.

Lemma 2. Every graph $G \in \mathcal{F}_e(3,3;5;n)$, for $n \geq 14$, contains an independent set of order 3. ■

We say that a graph G is $(+e, H)$ maximal if and only if $H \nsubseteq G$ and $H \subseteq G + e$, for every edge $e \in E(\overline{G})$. For graphs H and G and vertex set $S \subseteq V(G)$ we say that S is $(G, +v, H)$ maximal if and only if $H \nsubseteq G[S]$ and $H \subseteq G[S \cup \{v\}]$, for all $v \in V(G) \setminus S$.

Lemma 3. For every graph B obtained from a graph A by removing three vertices of some independent set of order 3 in A, where $A \in \mathcal{F}_e(3,3;5;n)$, $n \geq 14$ and A is $(+e, K_5)$ maximal, we have

(1) $K_5 \nsubseteq B$

(2) $\chi(B) \geq 5$

(3) For every edge $\{v,w\} \in E(\overline{B})$ there are vertices $a, b \in V(B)$ such that $B[\{v,w,a,b\}]$ is isomorphic to $K_4 - e$.

(4) For every vertex $u \in V(A) \setminus V(B)$ the set $N_A(u)$ is $(B, +v, K_4)$ maximal.

Proof. Property (1) is a consequence of $A \in \mathcal{F}_e(3,3;5)$. Property (2) is a consequence of Lemma 1. Properties (3) and (4) follow from the assumption that A is $(+e, K_5)$ maximal. ■

Lemmas 1, 2 and 3 guarantee that the following Algorithm A_1 generates all graphs in the set $\mathcal{F}_e(3,3;5;14)$ which are $(+e, K_5)$ maximal.

Algorithm A_1

Step 1: Start with the set $\mathcal{A} = \emptyset$, and the set $\mathcal{B}$ of all nonisomorphic graphs of order 11.

Step 2: Remove from $\mathcal{B}$ all graphs which do not fulfil conditions (1), (2) or (3) of lemma 3.

Step 3: For every graph $B \in \mathcal{B}$ find the family $\mathcal{M}_B = \{S \subseteq V(B) : S$ is $(B, +v, K_4)$ maximal$\}$. The set $\mathcal{M}_B$ contains potential neighborhoods of vertices of $(+e, K_5)$ maximal graphs in $\mathcal{F}_e(3,3;5;14)$. For every triple $S_1, S_2, S_3 \in \mathcal{M}_B$ construct a graph $A(B, S_1, S_2, S_3)$ by adding vertices v_1, v_2, v_3 to B so that $N_A(v_i) = S_i$ for $i = 1,2,3$. If $\chi(A) \geq 6$ and A is $(+e, K_5)$ maximal then add graph A to the set $\mathcal{A}$.

Step 4: For every graph $A \in \mathcal{A}$ if $A \to (3,3)^e$ does not hold then remove A from $\mathcal{A}$.

3 Results

The algorithm outlined above did not produce any graph $A \rightarrow (3,3)^e$, i.e. $\mathcal{A} = \emptyset$. A large number of graphs were produced in steps 2 and 3. These were generated by two independent implementations by the first two authors, and they always agreed up to isomorphism. The results of computations lead to the main theorem.

Theorem 1. $F_e(3,3;5) = 15$.

Proof. The above computations proved that $F_e(3,3;5) \geq 15$. Examples of $(3,3;5;15)^e$ graphs were already presented in [9], [18], [22]. ■

We say that graph G has *Sperner's property* if and only if for every pair of distinct vertices v, w, their neighborhoods are not contained in each other; i.e., $N_G(v) \nsubseteq N_G(w)$ and $N_G(w) \nsubseteq N_G(v)$.

The Algorithm A_2, which is a slight modification of Algorithm A_1, enabled us to generate all critical Folkman graphs $\mathcal{F}_e(3,3;5;15)$ which are $(+e, K_5)$ maximal.

Algorithm A_2

Apply the following modifications to Algorithm A_1

1. In Step 1 the starting set $\mathcal{B}$ is the set of all nonisomorphic graphs of order 12.

2. At the end of Step 3, in addition to conditions $\chi(A) \geq 6$ and that A is $(+e, K_5)$ maximal, it is required that the graph A has Sperner's property (since if A doesn't have Sperner's property and $A \rightarrow (3,3)^e$, then there would be a graph $(A - v) \rightarrow (3,3)^e$ contradicting Theorem 1)

There are 165 091 172 592 nonisomorphic graphs of order 12. After the second step, 217 524 627 graphs remain. Step 3 produces 299 543 761 graphs (including isomorphs), and 324 graphs remain after step 4. Among these 324 graphs only 19 are nonisomorphic. Thus we have:

Theorem 2. Up to isomorphism there are 19 $(+e, K_5)$ maximal $(3,3;5;15)^e$ graphs. ■

Using a simple algorithm which removes edges and tests whether the obtained graph has the property $\rightarrow (3,3)^e$ we were able to generate the whole set $\mathcal{F}_e(3,3;5;15)$ starting from the set of 19 graphs as in Theorem 2. The most interesting properties of the graphs $(3,3;5;15)^e$ are presented in the next theorem, where α denotes the order of the largest independent set and δ and Δ denote minimum and maximum degree:

Theorem 3. $|\mathcal{F}_e(3,3;5;15)| = 659$. For every graph $G \in \mathcal{F}_e(3,3;5;15)$:
(a) $56 \leq |E(G)| \leq 68$ (see Table 2)

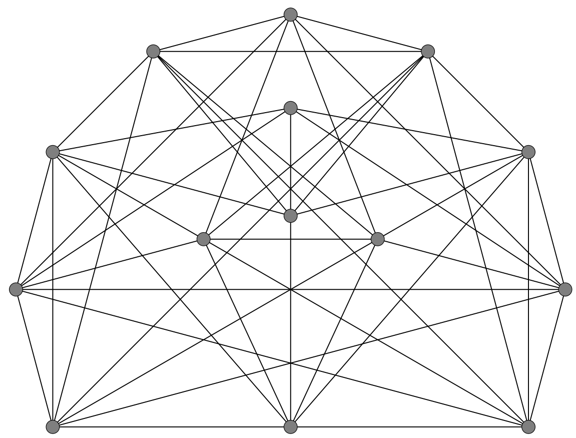

Figure 1. The unique bicritical $(\mathbf{3}, \mathbf{3}; \mathbf{4}; \mathbf{14})^v$graph.

(b) $\chi(G) = 6$
(c) $4 \leq \alpha(G) \leq 7$
(d) $5 \leq \delta(G) \leq 7$
(e) $11 \leq \Delta(G) \leq 14$ and $\Delta(G) \neq 13$ ■

The fact that there are no $(3, 3; 5; 15)^e$ graphs with $\Delta(G) = 13$ is easy to explain since such a graph would violate Sperner's property.

It is interesting that the graph presented in [18] has the smallest possible number of edges among $(3, 3; 5; 15)^e$ graphs. This is also the only $(3, 3; 5; 15)^e$ graph with $\alpha(G) = 7$. It is a subgraph of 18 graphs from $\mathcal{F}_e(3, 3; 5; 15)$, including itself.

We found that there exists exactly one *bicritical* graph in $\mathcal{F}_e(3, 3; 5; 15)$, for which deletion and addition of any edge destroys the Ramsey property or creates a K_5, respectively. It contains a vertex of degree 14. After removing this vertex we obtain the unique bicritical $(3, 3; 4; 14)^v$ graph, which is presented in Figure 1. This graph has only one nontrivial automorphism, contrary to our intuition which tells us that if there is a small number of some exceptional graphs they tend to have large groups of symmetry. Other graphs from $\mathcal{F}_e(3, 3; 5; 15)$ also have surprisingly small automorphism groups. 501 of them have only the trivial automorphism, and 132 graphs have only two automorphisms. The order of the automorphism group is less or equal to 14 for every $(3, 3; 5; 15)^e$ graph.

Let us remark that the inequality $\alpha(G) \geq 4$ for $G \in \mathcal{F}_e(3, 3; 5; 15)$ is not trivial. Knowing it we would be able to reduce the time of computation aproximately100 times starting with graphs of order 11 instead of 12 in Step 1 of Algorithm A_2, and then building graphs by adding quadruples of vertices in step 3.

Table 2 lists the numbers of critical $(3,3;5)^e$ graphs $c^e_m = |\{G \in \mathcal{F}_e(3,3;5;15) : |E(G)| = m\}|$, for all possible m.

Table 2. Statistics for the critical $(3,3;5)^e$ graphs.

m	56	57	58	59	60	61	62	63	64	65	66	67	68
c^e_m	1	2	7	20	39	58	80	119	144	111	58	17	3

Lemma 4. $F_e(3,3;5) \leq F_v(3,3;4) + 1$

Proof. It is enough to observe that if $G \in \mathcal{F}_v(3,3;4;n)$ then $H \in \mathcal{F}_e(3,3;5;n+1)$, where H is obtained by adding a vertex of degree n to graph G. ■

Several authors used the latter implication for the construction of $(3,3;5;15)^e$ graphs. Now it is helpful in establishing the exact value of $F_v(3,3;4)$.

Theorem 4. $F_v(3,3;4) = 14$

Proof. From Lemma 4 and Theorem 1 we have $F_v(3,3;4) \geq 14$. Examples of $(3,3;4;14)^v$ graphs were already presented in [18], [22]. ■

The result in the next theorem was somewhat unexpected.

Theorem 5. $\mathcal{F}_v(3,3;4;14) = \{G - v : G \subseteq \mathcal{F}_e(3,3;5;15), \deg_G(v) = 14\}$.

Proof. The argument used in the proof of Lemma 4 implies that

$$\mathcal{F}_v(3,3;4;14) \subseteq \{G - v : G \subseteq \mathcal{F}_e(3,3;5;15), \deg_G(v) = 14\}.$$

There are 153 graphs $(3,3;5;15)^e$ with one vertex of degree 14, and there is no $(3,3;5;15)$ graph with more than one vertex of degree 14. Using computers we checked that for each of these 153 cases, after deleting the vertex of degree 14, we obtain a $(3,3;4;14)^v$ graph. ■

Table 3 lists the numbers of critical $(3,3;4)^v$ graphs $c^v_m = |\{G \in \mathcal{F}_v(3,3;4;14) : |E(G)| = m\}|$, for all possible m.

Table 3. Statistics for the critical $(3,3;4)^v$ graphs.

m	42	43	44	45	46	47	48	49	50
c^v_m	1	2	7	20	37	45	28	11	2

4 Computations

Three powerful programs *nauty*, *makeg*, and *autoson*, implemented by Brendan McKay [15],[16] were used in our work. All of them were extensively tested and employed by many researchers in a variety of projects. We used *nauty* for fast isomorph rejection and finding automorphism groups, *makeg* to generate all nonisomorphic graphs on 11 and 12 vertices in algorithms A_1 and A_2, and *autoson* for distributing jobs over the network of computers. The entire project, run on 100+ computers of distinct architectures, used an equivalent of about 3 cpu years of a Sparc 2 SUN computer. All the algorithms specific for this project were implemented independently by two authors, and then a very large number of intermediate and final graphs were tested for isomorphism between the two implementations. The computational effort to prove $F_e(3,3;5) = 15$ and $F_v(3,3;4) = 14$ was only about 1% of the total cpu used; most of the time was needed for the complete enumeration of all critical $(3,3;5)^e$ graphs.

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References

[1] J. Bukor, A note on the Folkman number $F(3,3;5)$, *Math. Slovaca* **44** (1994), 479–480.

[2] P. Erdős, A. Hajnal, Research problem 2-5, *J. Combin. Theory* **2** (1967), 104.

[3] M. Erickson, An upper bound for the Folkman number $F(3,3;5)$, *J. Graph Theory* **17** (1993), 679–681.

[4] J. Folkman, Graphs with monochromatic complete subgraphs in every edge coloring, *SIAM J. Appl. Math.* **18** (1970), 19–24.

[5] P. Frankl, V. Rödl, Large triangle-free subgraphs in graphs without K_4, *Graphs and Combinatorics* **2** (1986), 135–144.

[6] R. L. Graham, On edgewise 2-colored graphs with monochromatic triangles and containing no complete hexagon, *J. Combin. Theory* **4** (1968), 300.

[7] R. L. Graham, J. H. Spencer, On small graphs with forced monochromatic triangles, Recent Trends in Graph Theory, *Lecture Notes in Math.* **186** (1971), 137–141.

[8] N. Hadziivanov, N. Nenov, On the Graham-Spencer number (in Russian), *C.R. Acad. Bulg. Sci.* **32** (1979), 155–158.

[9] N. Hadziivanov, N. Nenov, On Ramsey graphs (in Russian), *God. Sofij. Univ. Fak. Mat. Mech.* **78** (1984), 211–214.

[10] N. Hadziivanov, N. Nenov, Every (3,3)-Ramsey graph without 5-cliques has more than 11 vertices (in Russian), *Serdica* **11** (1985), 341-356.

[11] R. W. Irving, On a bound of Graham and Spencer for graph-coloring constant, *J. Combin. Theory* **15** (1973), 200–203.

[12] S. Lin, On Ramsey numbers and K_r-coloring of graphs, *J. Combin. Theory Ser. B* **12** (1972), 82–92.

[13] T. Łuczak, S. Urbański, A note on restricted vertex Ramsey numbers, *Periodica Mathematica Hungarica* **33 (2)** (1996), 101-103.

[14] T. Łuczak, A. Ruciński, S. Urbański, On minimal vertex Folkman graphs, Proceedings of *the Third Kraków Conference on Graph Theory, Kazimierz 97*, submitted.

[15] B. D. McKay, *Nauty User's Guide* (version 1.5), Technical report TR-CS-90-02, Computer Science Department, Australian National University, 1990.

[16] B. D. McKay, *nauty*, *makeg* and *autoson* documentation and source code, `http://cs.anu.edu.au/people/bdm/`.

[17] N. Nenov, A new lower bound for the Graham–Spencer number (in Russian), *Serdica* **6** (1980), 373–383.

[18] N. Nenov, An example of a 15-vertex (3,3)-Ramsey graph with clique number 4 (in Russian), *C.R. Acad. Bulg. Sci.* **34** (1981), 1487–1489.

[19] N. Nenov, On $(3,4)$ Ramsey graphs without 9-cliques (in Russian), *Annuaire Univ. Sofia Fac. Math. Inform.* **85** (1991), no. 1-2, 71-81.

[20] M. Schäuble, Zu einem Kantenfarbungsproblem. Remerkungen zu einer Note von R. L. Graham. *Wiss. Z. Techn. Hochsch. Ilmenau* **15** (1969), Heft 2 55–58.

[21] J. Spencer, Three hundred million points suffice, *J. Combin. Theory Ser. A* **49** (1988), 210–217. Also see erratum in Vol. **50**, p. 323.

[22] S. Urbański, Remarks on 15-vertex $(3,3)$-Ramsey graphs not containing K_5, *Disc. Math. Graph Theory* **16** (1996), 173-179.