# The First Classical Ramsey Number for Hypergraphs Is Computed

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## OVERVIEW

With the help of the computer, we have shown that in any coloring with two colors of the triangles on a set of 13 points there must exist a monochromatic tetrahedron. This proves the new upper bound $R(4,4;3) \le 13$. The previous best upper bound of 15 was derived independently by Giraud (1969 [2]), Schwenk (1978 [5]) and Sidorenko (1980 [6]). The first construction of a $R(4,4;3)$-good hypergraph on 12 points was presented by Isbell (1969 [3]), and the same one again more elegantly by Sidorenko (1980 [6]). We have constructed more than 200,000 $R(4,4;3)$-good hypergraphs on 12 points, but probably not the full set. $R(4,4;3)=13$ is the first known exact value of a classical Ramsey number for hypergraphs.

The solution was achieved with the help of a variety of algorithms relying on a strong connection between the colorings with two colors of the triangles on $n$ points and the so-called Turán set systems $T(n,5,4)$. The main criterion used to prune the search space for $R(4,4;3)$-good hypergraphs was to count the number of 4-sets containing two triangles of each color; such families of 4-sets are known to form Turán systems and their cardinalities must be minorized by the corresponding Turán numbers $T(n,5,4)$. We used an innovative method for generating large families of set systems which efficiently prevents isomorphic copies of set systems being produced. This method has many potential applications to other general computer searches for elusive combinatorial configurations. As a check on the correctness of the algorithms, many of the intermediate subfamilies of $R(4,4;3)$-good hypergraphs were generated by two different methods: from colorings of triangles on a smaller number of points and independently via Turán systems. An important component of the software used was a general set-system automorphism group program [4].

## 1. THEORETICAL BACKGROUND

The *Ramsey number* $R(k,l;s)$ is defined to be the least $n$ such that, in any coloring with two colors of the $s$-subsets of a set of $n$ elements, there is a $k$-subset all of whose $s$-subsets have the first color or there is an $l$-subset all of whose $s$-subsets have the second color.

The numbers $R(k,l)=R(k,l;2)$ are known as Ramsey numbers for standard graphs. There is an obvious interpretation of a coloring of all the $s$-subsets of an $n$-set with two colors as an $s$-uniform hypergraph: the edges are just the $s$-subsets of the first color. An $s$-uniform hypergraph $G$ is called $R(k,l;s)$-*good* if it has no set of $k$ vertices all of whose $s$-subsets are edges, and no set of $l$ vertices none of whose $s$-subsets are edges. The aim of this paper is to describe the theoretical background and the main ideas of the algorithms which led to the evaluation of the first nontrivial Ramsey number with $s>2$, namely $R(4,4;3)=13$.

A *Turán set system* $T(n,l,k)$ is a $k$-uniform hypergraph such that any subset of $l$ vertices contains at least one edge. The *Turán number* $T(n,l,k)$ is the minimal number of edges of any Turán set system $T(n,l,k)$ (the context will clarify this abuse of notation). The known nontrivial values of $T(n,l,k)$ are rare, but more abundant than known Ramsey numbers. There is an intimate relation between our problem and $T(n,5,4)$; the paper [1] is about the Turán numbers $T(n,5,4)$. We will use the abbreviation $T(n)$ for $T(n,5,4)$, in both its meanings. The numbers $T(n)$ and all minimal systems $T(n)$ are known for $n \le 10$ [1,6]; in particular $T(8)=14$, $T(9)=30$ and $T(10)=50$. We also easily have [1,6]:

$$T(n) \ge \lceil T(n-1)n/(n-4) \rceil . \qquad (1)$$

$W(n)$ will denote the class of 3-uniform hypergraphs on $n$ points, and let

$$R(n) = \{G \in W(n) : G \text{ is } R(4,4;3)\text{–}good\}.$$

Obviously $R(4,4;3) = \min\{n : R(n) = \emptyset\}$. Also, for each $G \in W(n)$ let $Fe(G)$ denote the family of 4-sets of points which contain an even number of edges (called *blocks*) of $G$, and let $fe(G)$ be the number of 4-sets in family $Fe(G)$. The theorem below establishes a fundamental link between Turán systems and our problem.

**Theorem 1** [2,5,6]: *If* $G \in W(n)$ *then* $S = Fe(G)$ *is a* $T(n)$ *system and* $S$ *has the property that every 5-set of vertices contains an odd number of blocks in* $S$ (*i.e.* 1,3 *or* 5).

This prompts two definitions.

**Definition 1:** $TC(n)$ *is the subclass of* $T(n)$ *formed by the systems* $S \in T(n)$ *such that every set of 5 vertices contains an odd number of blocks of* $S$.

**Definition 2:** (*TR Turán-Ramsey systems*)

$$TR(n) = \{S \in T(n) : S = Fe(G) \text{ for some } G \in R(n)\}.$$

The above also defines naturally, as for $T(n)$ systems/numbers, the corresponding numbers $TC(n)$ and $TR(n)$, where $TR(n)$ is undefined for $n \ge R(4,4;3)$. We obviously have $TR(n) \subseteq TC(n) \subseteq T(n)$ and $TR(n) \ge TC(n) \ge T(n)$, for systems and numbers, respectively. It has been conjectured (see [1]) that $TC(n) = T(n)$ holds for all $n$, and that the minimum set systems in $T(n)$ and $TC(n)$ are identical. Inequality (1) holds also for $TR(n)$ and $TC(n)$ numbers. We found that the minimum set systems for $T(n)$, $TC(n)$ and $TR(n)$ are identical for $n \le 10$.

**Fact 1:** ([6] and this work) *If* $S \in TC(n)$ *and* $b = |S|$ *then* $b$ *is even for* $n = 1,3$ *mod* 8, *and* $b$ *is odd for* $n = 5,7$ *mod* 8. *If* $G \in R(n)$ *and* $b = |G|$ *then* $b + fe(G)$ *is even for* $n = 0,2$ *mod* 8, *and* $b + fe(G)$ *is odd for* $n = 4,6$ *mod* 8.

Using Theorem 1, Fact 1 and (1) we obtain the following lower bounds for $TC(n)$:

**Theorem 2:** [6]

| $n$ | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|
| $TC(n) \ge$ | 80 | 120 | 175 | 245 | 335 |

Let $C(n,k)$ denote the binomial coefficient. Each of the three proofs of $R(4,4;3) \le 15$ [2,5,6] uses some fact equivalent to the next theorem:

**Theorem 3:** *If* $G \in R(n)$ *then*

$$fe(G) \le U(n) = C(n,4) - n \lceil (n-1) \cdot \lfloor (n-2)/2 \rfloor \cdot \lfloor (n-5)/2 \rfloor /6 \rceil .$$

Theorem 3 gives the following upper bounds $U(n)$ for $fe(G)$:

| $n$ | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $U(n)$ | 5 | 15 | 21 | 38 | 54 | 90 | 110 | 159 | 195 | 273 | 315 |

For any $G \in R(n)$ we obviously have $TC(n) \le TR(n) \le fe(G) \le U(n)$, thus Theorems 2 and 3 imply $R(4,4;3) \le 15$. Our aim is to find $R(13)$ and $R(14)$. Since $R(13)$ turns out to be empty, we will infer that $R(14)$ is too, and consequently $R(4,4;3)=13$.

Let $R(n,f,t)$ be the set of all $R(n)$ systems $G$ with $t$ triangles such that $f = fe(G)$, and $R(n,f) = \bigcup R(n,f,t)$. We also define the classes $T(n,f)$, $TC(n,f)$ and $TR(n,f)$, where $f$ fixes the number of 4-sets in the system. The unique hypergraph in $R(6,15,10)$ is the well known unique $2-(6,3,2)$ design, which can be considered as a $R(K_4^3 - t, K_4^3 - t;3)$-good hypergraph, where $K_4^3 - t$ is a tetrahedron without one triangle. Observe that $U(7)=21<35=C(7,4)$, thus we have $R(K_4^3 - t, K_4^3 - t;3)=7$. Our calculations also imply that $R(K_4^3 - t, K_4^3;3)=8$. It can be proved by hand [5] that $TR(8,15)$ and $TR(8,16)$ are empty. In general, $S \in TR(n)$ can have many nonisomorphic colorings in $R(n)$, i.e. there can be many systems $G \in R(n)$ such that $Fe(G)=S$.

We also prove the following:

**Theorem 4:** *If $G \in R(n,U(n))$ and $n=4p+q$ then $G$ is a $2-(n,3,K)$ design, i.e. every pair is covered by $k$ triangles for some $k \in K$, where*

$K = \{2p\text{-}2, 2p\text{-}1, 2p\}$ for $q=0$,
$K = \{2p\text{-}1, 2p\}$ for $q=1$,
$K = \{2p\}$ for $q=2$,
$K = \{2p, 2p+1\}$ for q=3.

## 2. ALGORITHMS

Consider the following diagram:

$$\begin{tikzcd}
TC(n) \arrow[r, dash, "4\to"] \arrow[d, dash] & TC(n+1) \arrow[d, dash] \\
R(n) \arrow[r, dash, "1\to"'] & R(n+1)
\end{tikzcd}$$

An arrow between two classes $F_1 \to F_2$ labeled *op* means that for each $G \in F_1$ the operation *op* produces all systems $H \in F_2$ naturally associated with $G$. The actual algorithms consider some subsets of these classes, for example $TR(n,f)$ or $R(n,f,t)$. The meaning for each input hypergraph $G$ and the difficulty of efficient implementation of each operations is as follows:

1 - *ramexp*, given $f$ and $G \in R(n)$ find its extensions in $R(n+1, \le f)$, hard
2 - *turpaint*, given $G \in TC(n)$ find systems $H \in R(n)$ such that $Fe(H)=G$, hard
3 - *ramtur*, given $G \in R(n)$ calculate $Fe(G)$, easy
4 - *turexp*, given $f$ and $G \in TC(n)$ find its extensions in $TC(n+1, \le f)$, hard

An important component of the software used was a general set-system automorphism group program *nauty* [4]. *Nauty* permits efficient detection of isomorphic set systems and, even more importantly, can be applied in a manner which permits generation of nonisomorphic set systems with only very limited explicit isomorphism testing. For various values of $n$, many subfamilies of $R(n)$ and $TR(n)$ have been computed several times, using totally disjoint paths in the diagram above. This provided a excellent check on the computations.

## 3. OUTLINES OF COMPUTATIONS

Outline I:

0. $R(4)$
1. $R(5)$
2. $R(6)$
3. $R(7,f)$, for $f = 7, 9$
4. $R(8,f)$, for $f = 14, 17, 18$
5. $R(9,f)$, for $f = 30, 32, 34$
6. $R(10,f)$, for $50 \le f \le 57$
7. $R(11,f)$, for $f = 84, 86, 88, 90$
8. $R(12,f)$, for $126 \le f \le 135$
9. $R(13)$
10. $R(14)$

Outline I, if done, would solve the entire problem. The data of step $i$ is sufficient to obtain the data of step $(i+1)$ by *ramexp*. The correctness of the above bounds for $f$ is a consequence of (1) for $TR$-systems, Fact 1, Theorem 3, known values of $T(n)$ for $n \le 10$, and $TC(11)$=84, $TC(12)$=126, where the last two equalities were obtained by *turexp* algorithm. For example, the equalities

$$\lceil 34 \cdot 10/6 \rceil = 57 \text{ and } \lceil 36 \cdot 10/6 \rceil = 60$$

show that the data of step 5 is sufficient and necessary to obtain $R(10,f)$ for $f \le 59$. The three $R(4)$ systems are easily constructed by hand. Steps 1-4 were done directly by *ramexp*, and indirectly by *ramtur*/*turexp*/*turpaint* algorithms following the path

$$R(4) \to TR(4) \to TR(5) \to TC(6) \to TC(7,\le 9) \to TC(8,\le 18) \to R(8,\le 18).$$

Both yielded the same 33539 $R(8,\le 18)$ systems. Unfortunately, not all further steps are feasible by this approach, since there are too many systems on the way. For example we estimate $|R(10,57)|$ to be at least of order $10^7$. Hence steps 5-9 of Outline I were replaced by Outlines II and III.

Outline II:

5. $R(9,f)$, for $f = 30, 32$
6. $R(10,f)$, for $50 \le f \le 56$
7. $R(11,f)$, for $f = 84, 86, 88$,
   and those $R(11,90)$ systems which are extensions of systems found at step 6
8. $R(12,f)$, for $126 \le f \le 134$,
   and those $R(12,135)$ systems which are extensions of systems found at step 7
9. $R(13,f)$, for $f =< 193$,
   and those $R(13,195)$ systems which are extensions of systems found at step 8

The correctness of the bounds on $f$ follows as for Outline I. Outline II has been completed directly by *ramexp* without using the lower bounds on $f$ (thus verifying them), and large portions of it were checked by an indirect approach via $TR$-systems. No $R(13)$ system was found.

**Theorem 5:** *Let $G$ be any system in $R(13)$ not obtained by Outline II and let $S$ range over the induced subsystems of $Fe(G)$. Then $G$ and $S$ satisfy all of the following:*

P0. $G \in R(13,195)$.

P1. $Fe(G)$ is a $2-(13,4,15)$ design.

P2. $G$ is a $2-(13,3,\{5,6\})$ design.

P3. In any induced subsystem of $G \in R(n)$, each pair is covered by at least $n-8$ and at most 6 triangles.

P4. Each triple is covered by at most 3 blocks of $S$.

P5. If $S \in TR(8)$ then $S$ has at least 17 blocks, and at least one $S \in TR(8)$ has at most 18 blocks.

P6. If $S \in TR(9)$ then $S$ has at least 34 blocks and at most 44 blocks, and at least one $S \in TR(9)$ has 34 blocks.

P7. If $S \in TR(10)$ then $S$ has at least 57 blocks and at most 60 blocks, and at least one $S \in TR(10)$ has 57 blocks.

P8. If $S \in TR(11)$ then $S$ has 90 blocks.

P9. If $S \in TR(12)$ then $S$ has 135 blocks.

Outline III:

Compute all $R(13)$ systems not obtained by Outline II.

Algorithms for Outline III can be easily devised by enforcing properties P0-P9 of Theorem 5. Full direct ($R(n) \rightarrow R(n+1)$) and indirect (via $TR$-systems) computations have been completed, and all intermediate results agreed. No $R(13)$ system was found. Hence we have:

**Theorem 6:** $R(4,4;3)=13$.

The above and other computations produced more than 200,000 nonisomorphic $R(12,f,t)$ systems with $f$ ranging from $TR(12)=126$ to $U(12)=159$, and $t$ ranging from 104 to 116. The total amount of computation used was of the order 6E13 machine instructions carried out on different computers in Canberra and Rochester.

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