Use of Max-Cut Algorithms in a Ramsey Arrowing Problem

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History of $F_e(3,3;4)$

What is the smallest order n of a K_4-free graph that is not a union of two triangle-free graphs?

1967	Erdős and Hajnal introduce the problem
1970	Folkman proves existence theorem
1975	Erdős offers \$100 for proving whether or not $F_e(3,3;4) < 10^{10}$
1988	Spencer gives a probabilistic proof for the bound 3×10^8. One year later, Hovey finds mistake and shows the bound to be 3×10^9
2007	Lu: $F_e(3,3;4) \leq 9697$
2008	Dudek and Rödl: $F_e(3,3;4) \leq 941$
2012	This work: $F_e(3,3;4) \leq 786$

Folkman Graphs and Numbers

For graphs F, G, H and positive integers s, t

- $F \to (s,t)^e$ iff for every 2-coloring of the edges F, there is a monochromatic K_s in the first color or K_t in the second
- $F \to (G,H)^e$ iff for every 2-coloring of the edges of F, there is a copy of G in the first color or a copy of H in the second

edge Folkman graphs

$\mathcal{F}_e(s,t;k) = \{G \to (s,t)^e,\ K_k \not\subseteq G\}$

edge Folkman numbers

$F_e(s,t;k) =$ the smallest n such that an n-vertex graph G is in $\mathcal{F}_e(s,t;k)$

Theorem: (Folkman 1970)

For all $k > \max(s,t)$, $F_e(s,t;k)$ and $F_v(s,t;k)$ exist.

Relation to Ramsey Numbers

$R(s,t) = \min\{n \mid K_n \to (s,t)^e\}$

Counting Triangles

For any blue-red coloring of graph G,

- $T_{\mathsf{BB}}(v)$, $T_{\mathsf{RR}}(v)$, and $T_{\mathsf{BR}}(v)$ counts triangles vuw where (v, u) and (v, w) are colored blue-blue, red-red, and blue-red
- T_{blue}, T_{red}, and T_{bluered} count the number of blue, red and blue-red triangles

Then,

- $\sum_{v \in V(G)} T_{\mathsf{BR}}(v) = 2T_{\mathsf{bluered}}$
- $\sum_{v \in V(G)} \big(T_{\mathsf{BB}}(v) + T_{\mathsf{RR}}(v)\big) = 3(T_{\mathsf{blue}} + T_{\mathsf{red}}) + T_{\mathsf{bluered}}$

$G \to (3,3)^e$ iff, for every coloring,

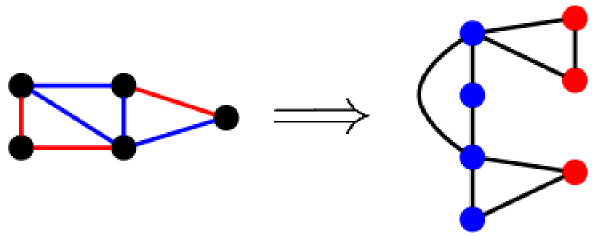
$$\sum_{v \in V(G)} T_{\mathsf{BR}}(v) < 2 \sum_{v \in V(G)} \big(T_{\mathsf{BB}}(v) + T_{\mathsf{RR}}(v)\big) \tag{1}$$

From Arrowing to Max-Cut

Define graph H

$$V(H) = E(G)$$
$$E(H) = \{(e,f) \mid e,f \in E(G),\ efg \text{ is a } \triangle \text{ in G for some edge } g\}$$

Then, $|V(H)| = |E(G)|$ and $|E(H)| = 3t_{\triangle}(G)$



Let $M(H)$ be the **Maximum Cut** of H.

Theorem: (Dudek and Rödl, 2008)

$$G \rightarrow (3,3)^e \quad \text{iff} \quad M(H) < 2t_{\triangle}(G) \tag{2}$$

Max-Cut Problem and Approximations

MAX CUT(H, k)
Given graph H and integer k, is there a cut $M_S(H)$ so that $M_s(H) \geq k$?

- One of Karp's original **NP**-complete problems (Karp 1972)

Based on this decision problem,

$$G \to (3,3)^e \quad \text{iff} \quad \textsc{MaxCut}\,(H, 2t_{\triangle}(G)) = \text{NO}$$

Can we approximate an upper bound to show arrowing?

Approach 1: Minimum Eigenvalue

Proposition: (Alon, 1996)

$$M(H) \leq \frac{|E(H)|}{2} - \frac{\lambda_{\text{min}}|V(H)|}{4}$$

Dudek-Rödl Technique

1. For graph G, construct graph H where $E(G) = V(H)$ and $E(H) = \{(e,f) \mid e,f \in E(G),\ efg \text{ is a } \triangle \text{ in G for some edge } g\}$
2. Let

$$\alpha = \frac{|E(H)|}{2} - \frac{\lambda_{\text{min}}(H)|V(H)|}{4}$$
$$\beta = 2t_{\triangle}(G)$$

3. If $\alpha < \beta$ then $G \to (3,3)^e$

$F_e(3,3;4) \leq 941$

Define circulant graph $G(n,r)$ as

- $V(G) = \mathbb{Z}_n$
- $E(G) = \{(x,y) | x - y = \alpha^r \bmod n\}$

Closeness $\rho = \frac{\alpha - \beta}{\alpha}$

n	r	ρ
127	3	0.0309
281	4	0.0423
457	4	0.0304
571	5	0.0441
701	5	0.0295
937	6	0.0485
941	**5**	**-0.0127**

This Work:

Graph with 860 vertices yields $\rho = -0.000056$

Approach 2: Goemans-Williamson Approximation

- Published in 1995
- Randomized approximation algorithm
- Expected value is at least $\alpha_{\mathrm{GW}} \approx .87856$ times the optimal value
  - First improvement on the $1/2$ constant from Sahni-Gonzales
- Relaxes the problem to a semidefinite program
  - First use of semidefinite programming in approximation algorithms
- Khot, Kindler and Mossel (2005): Assuming the Unique Games Conjecture and $P \neq NP$, Goemans-Williamson approximation algorithm is optimal

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Main Idea

Given graph with $V = \{1, \ldots, n\}$ and nonnegative weights $w_{i,j}$ for each pair of vertices (no edge $= 0$), we can write $M(G)$ as the integer quadratic program

$$\text{Maximize} \quad \frac{1}{2}\sum_{i<j} w_{i,j}(1 - y_i y_j) \tag{3}$$

$$\text{subject to:} \quad y_i \in \{-1, 1\} \quad \forall\, i \in V$$

Cut $S = \{i \mid y_i = 1\}$

We can relax some of the constraints of (3) and, specifically, extend the function to a larger space

- Extend y_i to $\mathbf{v}_i \in \mathbb{R}^n$ such that $\|\mathbf{v}_i\| = 1$
- Replace $y_i y_j$ with $\mathbf{v}_i \cdot \mathbf{v}_j$
- For matrix $Y = X^T X$, let $y_{ii} = 1$ and the ith column of $X = \mathbf{v}_i$.

New semidefinite program for symmetric matrix Y:

$$\begin{aligned} \text{Maximize} \quad & \frac{1}{2} \sum_{i<j} w_{i,j}(1 - y_{ij}) \qquad (4) \\ \text{subject to:} \quad & y_{ii} = 1 \quad \forall\, i \in V \\ & Y \succeq 0 \end{aligned}$$

The Algorithm

1. Solve (4) using an SDP solver (This is all we need!)
2. Decompose solution Y into $X^T X$ where $X = (\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n})$ using Cholesky decomposition
3. Choose random, uniformally distributed vector $\mathbf{r}$
4. $S = \{i \mid \mathbf{v}_i \cdot \mathbf{r} \geq 0\}$

$F_e(3,3;4) \leq 786$

Define graph $L(n,s)$ as follows:

- $V\big(\ L(n,s)\ \big) = \mathbb{Z}_n$
- $E\big(\ L(n,s)\ \big) = \big\{(u,v) \mid u \neq v$ and $u - v \equiv s^i \bmod n$ for some $i \in \{0,1,2,\ldots,m-1\}\big\}$, where m is the smallest positive integer such that $s^m \equiv 1 \bmod n$.

Let L_{786} be $L(785,53)$ with one additional vertex connected to 60 of the original vertices.

`SDPLR-MC`, `SDPLR`, `SBmethod`, and `SpeeDP` all give an upper bound of at most 857753.

$$M\left(H\left(L_{786}\right)\right) \leq 857753 < 2t_{\triangle}(L_{786}) = 857762.$$

Therefore, $L_{786} \rightarrow (3,3)^e$ and $F_e(3,3;4) \leq 786$.

Moving Forward

$G(127, 3)$

Conjecture: $G(127, 3) \rightarrow (3, 3)^e$

- Resilient to SAT, Dudek-Rödl and Goemans-Williamson
- Other techniques: MaxSAT approximation, simulated annealing?

Ideas:

- Adding edges to $G(127, 3)$
- Removing edges from $G(127, 3)$
- Embedding $G(127, 3)$ multiple times

Moving Forward

Minimum Eigenvalue vs. Goemans-Williamson

- Testing shows that Goemans-Williamson often provides better bounds
- However, MATLAB's `eigs` can handle larger instances
- Both can fail easy instances (all $F_e(3,3;5)$ graphs)

	MinEigs	SDP
K_6	Pass	Pass
$K_3 + C_5$	Fail	Fail
$K_4 + C_5$	Fail	Pass

Other Max-Cut methods?

- Directly solve integer program
- Rendl, Rinaldi, Wiegele: *Solving Max-Cut to Optimality by Intersecting Semidefinite and Polyhedral Relaxations*

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Thank you!