On Some Three-Color Ramsey Numbers for Paths

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Abstract

For graphs \(G_1, G_2, G_3\), the three-color Ramsey number \(R(G_1, G_2, G_3)\) is the smallest integer \(n\) such that if we arbitrarily color the edges of the complete graph of order \(n\) with 3 colors, then it contains a monochromatic copy of \(G_i\) in color \(i\), for some \(1 \leq i \leq 3\).

First, we prove that the conjectured equality \(R(C_{2n}, C_{2n}, C_{2n}) = 4n\), if true, implies that \(R(P_{2n+1}, P_{2n+1}, P_{2n+1}) = 4n + 1\) for all \(n \geq 3\). We also obtain two new exact values \(R(P_8, P_8, P_8) = 14\) and \(R(P_9, P_9, P_9) = 17\), furthermore we do so without help of computer algorithms. Our results agree with a formula \(R(P_n, P_n, P_n) = 2n - 2 + (n \text{ mod } 2)\) which was proved for sufficiently large \(n\) by Gyárfás, Ruszinkó, Sárközy, and Szemerédi in 2007. This provides more evidence for the conjecture that the latter holds for all \(n \geq 1\).

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1 Definitions

In this paper all graphs are undirected, finite and contain neither loops nor multiple edges. Let \( G \) be such a graph. The vertex set of \( G \) is denoted by \( V(G) \), the edge set of \( G \) by \( E(G) \), and the number of edges in \( G \) by \( e(G) \). For any edge coloring \( F \) of a complete graph, \( F^i \) will denote the graph induced by the edges of color \( i \) in \( F \). Let \( P_k \) (resp. \( C_k \)) be the path (resp. cycle) on \( k \) vertices. The circumference \( c(G) \) of a graph \( G \) is the length of its longest cycle.

The Turán number \( T(n, G) \) is the maximum number of edges in any \( n \)-vertex graph that does not contain any subgraph isomorphic to \( G \). A graph on \( n \) vertices is said to be extremal with respect to \( G \) if it does not contain a subgraph isomorphic to \( G \) and has exactly \( T(n, G) \) edges.

For given graphs \( G_1, G_2, \ldots, G_k \), \( k \geq 2 \), the multicolor Ramsey number \( R(G_1, G_2, \ldots, G_k) \) is the smallest integer \( n \) such that if we arbitrarily color the edges of the complete graph of order \( n \), \( K_n \), with \( k \) colors, then it contains a monochromatic copy of \( G_i \) in color \( i \), for some \( 1 \leq i \leq k \). A coloring of the edges of the \( K_n \) with \( k \) colors is called a \((G_1, G_2, \ldots, G_k; n)\)-coloring, if it does not contain a subgraph isomorphic to \( G_i \) in color \( i \) for any \( 1 \leq i \leq k \). In the diagonal cases of \( G = G_i \) we will write \( R_k(G) = R(G_1, G_2, \ldots, G_k) \). Finally, we will refer to the first three colors of such Ramsey colorings as red, blue and green, respectively.

2 Overview

In this article we study the values of three-color diagonal Ramsey numbers for paths. In the case of two color Ramsey numbers, a well known theorem of Gerencsér and Gyárfás [7] states that \( R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1 \) for \( n \geq m \geq 2 \).

Clearly, we trivially have \( R_3(P_1) = 1 \) and \( R_3(P_2) = 2 \). The cases \( R_3(P_3) = 5 \) and \( R_3(P_4) = 6 \) are easy but need some thought, while the results \( R_3(P_5) = 9 \), \( R_3(P_6) = 10 \) and \( R_3(P_7) = 13 \) already required help of computer algorithms (see section 6.4.1 of [13] for details and references to these and other related cases). The first open cases are those of \( R_3(P_8) \) and \( R_3(P_9) \), which are determined later in this paper. All known values agree with a very remarkable result obtained by Gyárfás, Ruszinkó, Sárközy, and Szemerédi in 2007 [10] formulated as follows.
Theorem 1 ([10]) For all sufficiently large $n$, we have

$$R_3(P_n) = \begin{cases} 
2n - 1 & \text{for odd } n, \\
2n - 2 & \text{for even } n.
\end{cases}$$  \hspace{1cm} (1)$$

The proof of Theorem 1 is very long and complicated. Our attempts to extract from it any reasonable bound on how large $n$ should be for (1) to hold, failed. Actually, Faudree and Schelp [6], already in 1975, stated that “they feel” that (1) holds for all $n$. They did so when considering more general cases of $R(P_m, P_n, P_k)$ for paths of different lengths. We believe that the diagonal case deserves the status of a conjecture.

Conjecture 2 ([6]) $R_3(P_n) = 2n - 2 + (n \mod 2)$ holds for all $n \geq 1$.

The appropriate critical colorings without monochromatic $P_n$ are known for all $n \geq 1$. For $n \geq 2$, one can obtain them by using a “blow-up” of factorization of $K_4$. The partition of the edges of $K_4$ into 3 matchings $2K_2$ in 3 distinct colors gives a witness coloring for $R_3(P_3) > 4$. For general odd $n = 2m - 1 \geq 5$, a witness coloring for $R_3(P_n) > 4m - 4$ can be obtained by blowing up each vertex of such colored $K_4$ into the sets of $m - 1$ vertices, and coloring the edges within the new 4 sets arbitrarily. Similarly for $n = 2m$, a witness coloring for $R_3(P_n) > 4m - 3$ can be obtained by blowing up three vertices of $K_4$ to $m - 1$ vertices, and one to $m$ vertices (for more details see [10]).

It is interesting to see (1) in the context of the conjectured values of three-color diagonal Ramsey numbers for cycles.

Conjecture 3 ([4][3])

$$R_3(C_n) = \begin{cases} 
4n - 3 & \text{for odd } n \geq 5, \\
2n & \text{for even } n \geq 6.
\end{cases}$$  \hspace{1cm} (2)$$

The odd case was conjectured by Bondy and Erdős in 1981 [4], while the even case by the second author in 2005 [3]. Like with (1) for paths, (2) is known to hold for all sufficiently large $n$. For the odd $n$ odd case, this result and an outline of the proof was described by Kohayakawa, Simonovits and Skokan in 2005 [11], but it took 8 years for the full proof to finally achieve the status ”to appear” [12]. The case for even $n$ was settled by Benevides
and Skokan in 2009 [1]. These results followed the exact asymptotic results obtained by Luczak and others (see also section 6.3.1 of [13] for details and references to other related cases). We know that (2) holds for all \( n \geq n_0 \) for some \( n_0 \), though there seems to be no easy way to find any reasonable upper bound on \( n_0 \). The first open cases of Conjecture 3 are those of \( R_3(C_9) \) and \( R_3(C_{10}) \).

In section 4 we will prove an interesting implication that the even \( n \) case of (2) implies the odd \( (n + 1) \) case of (1) for \( n \geq 6 \). The equalities \( R_3(C_6) = 12 \) [16] and \( R_3(C_8) = 16 \) [14] were obtained with the help of computer algorithms. Thus, it will imply that \( R_3(P_7) = 13 \) and \( R_3(P_9) = 17 \). We will also provide a computer-free proof of the latter. Finally, we prove that \( R_3(P_8) = 14 \), which leaves \( R_3(P_{10}) \) as the first open case of (1).

### 3 Related Background Results

Gyárfás, Rousseau and Schelp [9] completely solved the question of what is the maximum number of edges \( f(m, n, k) \) in any \( P_k \)-free subgraph of the complete bipartite graph \( K_{m,n} \). They also characterized all the corresponding extremal graphs. Tables III and IV in [9] present formulas for \( f(m, n, k) \) for even and odd \( k \), respectively, and Tables I and II therein describe the constructions of all the extremal graphs achieving \( f(m, n, k) \). In the proofs of sections 4 and 5 we will refer to these tables several times.

Also in the proofs we will need some values of Turán numbers for paths. In order to determine the required \( T(n, P_k) \), the following theorem by Faudree and Schelp [6], which enhances and condenses the results by Erdős and Gallai [5], will be used.

**Theorem 4 ([6][5])** If \( G \) is a graph with \( |V(G)| = kt + r \), \( r < k \), \( 0 \leq t, r \), containing no \( P_{k+1} \), then \( |E(G)| \leq t\binom{k}{2} + \binom{r}{2} \) with equality if and only if \( G \) is either \( tK_k \cup K_r \) or \( ((t - l - 1)K_k) \cup (K_{(k-1)/2} + K_{(k+1)/2+l} + lk + r) \) for some \( 0 \leq l < t \) when \( k \) is odd, \( t > 0 \), and \( r = (k \pm 1)/2 \).
The following notation and terminology comes from [2]. For positive integers \(a\) and \(b\), define \(r(a, b)\) as

\[
r(a, b) = a - b\left\lfloor \frac{a}{b} \right\rfloor = a \mod b.
\]

For integers \(n \geq k \geq 3\), define \(w(n, k)\) by

\[
w(n, k) = \frac{1}{2}(n-1)k - \frac{1}{2}r(k-r-1),
\]

where \(r = r(n-1, k-1)\).

Woodall's theorem [15] can then be formulated as follows.

**Theorem 5 ([2])** Let \(G\) be a graph on \(n\) vertices and \(m\) edges with \(m \geq n\) and circumference \(c(G)\) equal to \(k\). Then

\[
m \leq w(n, k),
\]

and this result is best possible.

In [2], one can find the description of all extremal graphs achieving \(w(n, k)\).

### 4 Progress on \(R_3(P_{2n+1})\)

First we prove the following general implication.

**Theorem 6** For all \(n \geq 3\), if \(R_3(C_{2n}) = 4n\), then \(R_3(P_{2n+1}) = 4n + 1\).

**Proof.** The lower bound follows from the “blow-up” construction commented on after the statement of Conjecture 2 in section 2 (see also [10]).

For the upper bound, suppose that there exists a 3-edge coloring of \(K_{4n+1}\) without monochromatic \(P_{2n+1}\). From the assumption that \(R_3(C_{2n}) = 4n\), we know that this coloring contains a monochromatic \(C_{2n}\). Without loss of generality, we assume that it is red. Now, in order to avoid red \(P_{2n+1}\) no vertex on this cycle can be connected by a red edge to any vertex outside of the cycle. Hence, we have a complete bipartite graph \(K_{2n,2n+1}\) with only blue
and green edges. Let the parts of this bipartite graph be called $X$ (vertices on the cycle) and $Y$ (vertices outside of the cycle). Using the notation of [9], we have

\[ a = 2n = |X|, \quad b = 2n + 1 = |Y|, \quad c = n - 1, \quad a = 2(c + 1), \]

hence, if we apply the last row of Table IV in [9], then we obtain

\[ f_1(a, b, c) = (a + b - 2c)c = 2n^2 + n - 3. \]

This implies that

\[ 2f_1(a, b, c) = 4n^2 + 2n - 6 < 2n(2n + 1) = |X||Y|, \]

and therefore blue and green edges cannot account for all the edges of $K_{X,Y}$ without creating a monochromatic $P_{2n+1}$. This completes the proof of the upper bound $R_3(P_{2n+1}) \leq 4n + 1$. \hfill \Box

**Corollary 7** $R_3(P_7) = 13$ and $R_3(P_9) = 17$.

**Proof.** It is known that $R_3(C_6) = 12$ [16] and $R_3(C_8) = 16$ [14]. By Theorem 6, these imply that $R_3(P_7) = 13$ and $R_3(P_9) = 17$. \hfill \Box

The upper bounds in $R_3(C_6) = 12$ and $R_3(C_8) = 16$ were obtained with the help of computer algorithms. In the proof of the next theorem we provide a computer-free proof of the upper bound $R_3(P_9) \leq 17$. The proof of $R_3(P_7) \leq 13$ can be obtained by a similar reasoning.

**Theorem 8** *(computer-free)*

\[ R_3(P_9) \leq 17. \]

**Proof.** We need to show that each 3-coloring of the edges of $K_{17}$ contains a monochromatic $P_9$. Let us suppose that there is a $(P_9, P_9, P_9; 17)$-coloring $G$ with colors red, blue and green, forming graphs $G^1$, $G^2$ and $G^3$, respectively. Since $K_{17}$ has 136 edges, we may assume without loss of generality that there are at least 46 red edges, i.e. $e(G^1) \geq 46$. 
Since by (3) we have $w(17, 6) = 46$, it follows by Theorem 5 that $G^1$ contains a cycle $C_k$ for some $k \geq 6$. One can easily verify that the critical graphs in this case [2] have $P_9$, and thus $k \geq 7$. If $k \geq 9$, we immediately obtain a red $P_9$, a contradiction. If $k = 8$, then to avoid a $P_9$ in $G^1$ we have a bipartite graph $G'$ with partite sets of order 8 and 9, respectively. In order to avoid monochromatic $P_9$ in $G^2$ and $G^3$, $G'$ contains at most 66 blue and green edges (use the last row of Table IV in [9]). Each of at most 6 other edges of $G'$ are red and together with the $C_8$ they contain a red $P_9$, again a contradiction. Hence, in the rest of the proof we will assume that $G$ has a red $C_7$ with vertices $C = \{c_1, c_2, ..., c_7\}$, and the remaining vertices are $P = \{p_1, p_2, ..., p_{10}\}$.

**Claim.** Let $H$ be a 3-coloring of the edges of $K_{17}$, and suppose that $P_9 \not\subseteq H^i$ for $1 \leq i \leq 3$, $|H^1| \geq 46$, $c(H^1) = 7$, and $H^1$ contains a $C_7$. Then there are at least 4 vertices in $V(H) \setminus V(C_7)$ joined by at least one red edge to the cycle $C_7$.

**Proof of the Claim.** Consider the coloring $H$ as stated above. Let the vertices of $C_7$ in $H^1$ be $C = \{c_1, c_2, ..., c_7\}$, and the remaining vertices of $H$ are $P = \{p_1, p_2, ..., p_{10}\}$. We will use the tables in [9] several times when considering bipartite subgraphs of $K_{|C|, k} = K_{7, k}$ for $k = 10, 9, 8$ and 7. We prove that there are red edges in these bipartite subgraphs, and so for each $k$ we obtain one more vertex in $V(H) \setminus C$ joined to the cycle $C_7$ by at least one red edge.

The maximum possible number of edges in a bipartite graph with partite sets 7 and $k = 10$ or 9 without $P_9$ is $7 + 3(k - 1)$, which follows from Table IV in [9] with $a = 7$, $b = k$, $c = 3$ and $f_1(a, b, c) = a + (b - 1)c$. Since $2(7 + 3(k - 1)) < 7k = e(K_{7, k})$ for $k = 10$ and 9, we obtain the first 2 vertices, say $p_1$ and $p_2$, connected to $C_7$ by at least one red edge.

Now, we consider the bipartite graph $K_{|C|, P \setminus \{p_1, p_2\}} = K_{7, 8}$. Similarly to the previous case, the maximum number of edges in this bipartite graph without $P_9$ is $7 + 3(8 - 1) = 28$. This time, however, this is exactly half of the edges of $K_{7, 8}$, so we need to consider the possible extremal graphs. By Table II in [9] these extremal graphs are $G_{14}$ and $G_{15}$ with $a = 7$, $b = 8$ and $c = 3$, and they can be eliminated as follows:

- $G_{14} = K_{7, 8} - K_{4, 7}$. Clearly, $K_{4, 7}$ contains a $P_9$, so it cannot consist of the edges of single color, a contradiction.

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$G_{15} = K_{4,4} \cup K_{3,4}$, which under bipartite complement is isomorphic to itself. Let us consider vertex $p_1$. To avoid red $C_8$ the vertex $p_1$ is joined by at most 3 red edges to the cycle $C_7$. By considering the remaining edges from the vertex $p_1$ we see that at least one, say blue, is connected to the $K_{4,4}$ part of $G_{15}$ in blue. This easily gives a monochromatic $P_9$, a contradiction.

Thus, we have the third vertex, say $p_3$, connected to $C$ by a red edge.

Finally, let us consider the bipartite graph $K_{C,P \setminus \{p_1,p_2,p_3\}}$. By the third case in Table IV of [9], the maximum possible number of edges in this bipartite graph without $P_9$ is $7 + 3(7-1) = 25$. By Table II, there are two possible extremal graphs: $G_{14}$ and $G_{15}$ which now are $K_{7,7} - K_{4,6}$ and $K_{4,4} \cup K_{3,3}$, respectively. By considering possible edges from $p_1$, $p_2$ and $p_3$ to the cycle $C_7$, similarly as for $k = 8$, we have a monochromatic $P_9$, so we obtain the fourth required vertex $p_4$. This completes the proof of the Claim.

We have $m \geq 4$ vertices $M = \{p_1, \ldots, p_m\} \subset P$ not on the red $C_7$ joined to it by some red edges. Note that any red edge $\{p_i, p_j\}$ prevents $p_i$ and $p_j$ to be connected by any red edge to $C$, hence $P$ induces at most $\binom{10-m}{2}$ red edges. To avoid red $C_8$, the vertices in $M$ can be joined by at most 3 red edges each to $C$ (to vertices nonadjacent on the cycle $C_7$).

First, consider the case when a vertex in $M$, say $p_1$, has 3 red edges to $C$, wolog $\{p_1,c_1\}$, $\{p_1,c_3\}$ and $\{p_1,c_5\}$. Note that no vertex $p \in M$, $p \neq p_1$, can be joined by any red edge to the vertices in the set $\{c_2,c_4,c_6,c_7\}$, since otherwise a red $P_9$ from $p$ to $p_1$ can be easily constructed. In addition, if there is the red edge $\{p_2,c_5\}$, then the edges $\{c_4,c_6\}$, $\{c_4,c_7\}$, $\{c_2,c_4\}$ are blue or green. For example, if $\{c_4,c_6\}$ is red, then $p_2c_5c_4c_6c_7c_1c_2c_3p_1$ is a red $P_9$. Similarly, if $\{p_2,c_1\}$ or $\{p_2,c_3\}$ is red, then at least three edges induced in $C$ must be blue or green. In all cases, $C$ induces at most 18 red edges. Thus, counting red edges in $C$, between $C$ and $P$, and in $P$, we have

$$e(G^1) \leq 18 + 3m + \binom{10-m}{2}. \quad (4)$$

Observe that the set $M \cup \{c_2,c_4,c_6,c_7\}$ induces only blue and green edges, hence $R(P_9, P_9) = 12$ [7] implies that $m + 4 \leq 11$. By the Claim we have $m \geq 4$, so $4 \leq m \leq 7$, and we find that $e(G^1) < 46$ for all possible $m$. This is a contradiction.
Finally we consider the case when all vertices in $M$ are connected to $C$ by at most 2 red edges. Counting again red edges, for all possible $4 \leq m \leq 10$, we obtain
\[ e(G^1) \leq \binom{7}{2} + 2m + \binom{10 - m}{2} < 46, \] (5)
which is a contradiction.

\section*{5 $R_3(P_8) = 14$}

We begin with a lemma which is technically very similar to the claim within the proof of Theorem 8.

\textbf{Lemma 9} Let $H$ be a 3-coloring of the edges of $K_{14}$, and suppose that $P_8 \not\subseteq H^i$ for $1 \leq i \leq 3$, $|H^1| \geq 31$, $c(H^1) = 6$, and $H^1$ contains a $C_6$. Then there are at least 3 vertices in $V(H) \setminus V(C_6)$ joined by at least one red edge to the cycle $C_6$.

\textbf{Proof.} We give only the sketch of proof because the details are very similar to those in the proof of Claim in Theorem 8. By using three times Tables I and III in [9] and considering bipartite subgraphs $K_{6,k}$ for $k = 8, 7, 6$, we obtain that the maximum number of edges in these bipartite subgraphs without $P_8$ is $3k$. From Table I, the extremal graphs are $K_{3,l} \cup K_{3,k-l}$, where $0 \leq l \leq k$. By considering the remaining edges of $H$, one can easily obtain a monochromatic $P_8$ in all cases, a contradiction. \hfill $\square$

\textbf{Theorem 10}

\[ R_3(P_8) = 14. \]

\textbf{Proof.} We need to show that every 3-edge coloring of $K_{14}$ contains a monochromatic $P_8$. Let us suppose that there is a $(P_8, P_8, P_8; 14)$-coloring $G$ with colors red, blue and green, forming graphs $G^1$, $G^2$ and $G^3$, respectively. Since $K_{14}$ has 91 edges, we may assume without loss of generality that there are at least 31 red edges, i.e. $e(G^1) \geq 31$.

Since by (3) we have $w(14, 5) = 31$, it follows by Theorem 5 that $G^1$ contains a cycle $C_k$ for some $k \geq 5$. One can easily verify that the critical
graphs in this case (see [2]) have $P_8$, and thus $k \geq 6$. If $k \geq 8$, then we immediately obtain a $P_8$, a contradiction. If $k = 7$, then to avoid a $P_8$ in $G^1$ we have a bipartite graph $G'$ with two partite sets of order 7. In order to avoid monochromatic $P_8$ in $G^2$ and $G^3$, the graph $G'$ contains at most 48 blue and green edges (use row 3 in Table III in [9]). At least one remaining edge of $G'$ is red and together with the $C_7$ we have a red $P_8$, a contradiction. Hence, in the rest of the proof we will assume that $G$ has a red $C_6$ with vertices $C = \{c_1, c_2, \ldots, c_6\}$, and the remaining vertices are $P = \{p_1, p_2, \ldots, p_8\}$.

By Lemma 9 we have $m \geq 3$ vertices $M = \{p_1, \ldots, p_m\} \subset P$ not on the red $C_6$ joined to it by some red edges. Note that any red edge $\{p_i, p_j\}$ prevents $p_i$ and $p_j$ to be connected by any red edge to $C$, hence $P$ induces at most $\binom{8-m}{2}$ red edges. To avoid red $C_7$, the vertices in $M$ can be joined by at most 3 red edges each to $C$ (to vertices nonadjacent on the cycle $C_6$). We will be counting red edges in $C$, between $C$ and $P$, and in $P$, similarly as in (4) and (5).

First, consider the case when all the vertices in $M$ are connected to $C$ by at most 2 red edges each. If at least one them is connected to 2 vertices in $C$, then at least one of the edges induced by $C$ is not red, or there are less than $2m$ edges between $C$ and $P$. Hence, for all possible $3 \leq m \leq 8$, we have

$$\begin{align*}
e(G^1) &\leq \binom{6}{2} + 2m - 1 + \binom{8-m}{2} < 31,
\end{align*}$$

which gives a contradiction.

The remaining case is when some vertex in $M$ is connected to $C$ by exactly 3 red edges, say $p_1$, and the red edges from $p_1$ to $C$ are $\{p_1, c_1\}$, $\{p_1, c_3\}$, $\{p_1, c_5\}$. Then no vertex $p_i \in P$, $2 \leq i \leq 8$, can be joined by a red edge to any of the vertices in the set $\{c_2, c_4, c_6\}$. In addition, if there is the red edge $\{p_2, c_1\}$, then the edges $\{c_2, c_1\}$, $\{c_2, c_3\}$, $\{c_4, c_6\}$ are blue or green. For example, if $\{c_2, c_3\}$ is red, then $p_2 c_1 c_2 c_4 c_3 p_1 c_5 c_6$ is a red $P_8$. Similarly, if $\{p_2, c_3\}$ or $\{p_2, c_5\}$ is red, then at least the same three edges induced in $C$ must be blue or green. In all cases, $C$ induces at most 12 red edges.

Observe that the set $M \cup \{c_2, c_4, c_6\}$ has only blue and green edges, hence $R(P_8, P_8) = 11$ [7] implies that $m + 3 \leq 10$. Note that if $m = 7$, then the sole vertex in $P \setminus M$ is not in any red edge, so we can decrease the range of $m$ further to $3 \leq m \leq 6$. This time we obtain

$$\begin{align*}
e(G^1) &\leq 12 + 3m + \binom{8-m}{2}.
\end{align*}$$

(6)
\( e(G^1) \) can achieve 31 in (6) for \( m = 3 \) and \( m = 6 \), furthermore only in cases when all (3 or 6) vertices in \( M \) are connected by exactly 3 red edges to \( C \). We will show that in both cases \( G \) has a blue or green \( P_8 \).

If \( m = 6 \), then the equality in (6) implies that \( P \cup \{c_2, c_4, c_6\} \) contains exactly one red edge between 2 vertices in \( P \setminus M \), or equivalently, the \( K_{11} - e \) with vertices \( P \cup \{c_2, c_4, c_6\} \) has all its 54 edges blue or green. By Theorem 4 with \( k = 7 \), \( t = 1 \) and \( r = 4 \) we obtain \( T(11, P_8) = 27 \). One can easily check that it is not possible for two copies of the corresponding extremal graphs to cover \( K_{11} - e \).

The last situation to consider is that of \( m = 3 \), where \( G^1 \) has two components: one spanned by 9 vertices of \( C \cup M \) with 21 red edges and a red \( K_5 \) on vertices \( Q = P \setminus M = \{p_4, \ldots, p_8\} \). The set \( H = M \cup \{c_2, c_4, c_6\} \) has no red edges. Denote by \( R \) the set \( \{c_1, c_3, c_5\} \). The 60 edges of \( G^2 \cup G^3 \) form a complete \( K_6 \) on \( H \) and two complete bipartite graphs \( K_{H, Q} \) and \( K_{Q, R} \). Let \( P \) be the longest monochromatic, say blue, path in \( H \), and denote by \( a \) and \( b \) its endpoints. By Theorem 4 we have \( T(6, P_4) = 6 \), which implies that \( l = 6 \) or \( l = 5 \). We have the following possibilities:

**Case 1.** There are no blue edges joining \( a \) or \( b \) to \( Q \) (for \( l = 5 \) or \( l = 6 \)).

We have \( H \cup R = C \cup M \), and let \( S = C \cup M \setminus \{a, b\} \). We consider the complete bipartite graph \( K_{5,7} \) with partite sets \( Q \) and \( S \). Because all the edges from \( a \) and \( b \) to \( Q \) are green, this \( K_{Q,S} \) cannot have green \( P_4 \). The third row of Table III in [9], with \( a = 5 \), \( b = 7 \) and \( c = 1 \), implies that there are at most 10 green edges between \( Q \) and \( S \). Clearly, \( K_{Q,S} \) cannot have blue \( P_8 \). We now use the second row of the same Table III with \( c = 3 \), and see that there are at most 21 blue edges between \( Q \) and \( S \). There are not enough green and blue edges to cover all 35 edges of \( K_{Q,S} \), which is a contradiction.

**Case 2.** There is a blue edge from \( a \) to \( Q \), say \( \{a, p_4\} \), and \( l = 6 \).

Let the blue \( P_1 \) in \( H \) be \( a s_1 s_2 s_3 s_4 b \). If there is no blue \( P_8 \), then all the edges joining \( b \) to \( p_i \), \( 5 \leq i \leq 8 \), and joining \( p_4 \) to \( R \) are green. We consider the colors of the edges from \( s_4 \) to the set \( Q \setminus \{p_4\} = \{p_5, p_6, p_7, p_8\} \). This case is now broken into three subcases, as follows:

1. There are at least two blue edges from \( s_4 \) to \( Q \setminus \{p_4\} \), say \( \{s_4, p_5\} \) and \( \{s_4, p_6\} \). To avoid blue \( P_8 \) all the edges between \( \{p_5, p_6\} \) and \( R \) must be green, but in this case we have a green \( P_8 = p_8b p_5c_1 p_6c_3 p_4c_5 \).
2. There is exactly one such blue edge, say \( \{s_4, p_5\} \). To avoid blue \( P_8 \) all the edges between \( p_5 \) and \( R \) must be green, but then we have a green \( P_8 = c_1p_4c_3p_5b_6s_4p_7 \).

3. All edges from \( s_4 \) to \( \{p_5, p_6, p_7, p_8\} \) are green. If there is a green edge between \( R \) and \( \{p_5, p_6, p_7, p_8\} \), say \( \{c_1, p_5\} \), then we have a green \( p_7s_4p_6b_5c_1p_4c_3 \). So, assume that all edges from \( R \) to \( \{p_5, p_6, p_7, p_8\} \) are blue. If there is at least one blue edge from \( \{p_5, p_6, p_7, p_8\} \) to \( \{s_2, s_3\} \), say \( \{p_5, s_2\} \), then we have a blue \( p_4s_1s_2p_5c_1p_6c_3 \). In the opposite case we obtain a green \( p_4s_4p_5s_3p_6s_2p_7b \).

\textbf{Case 3.} There is a blue edge from \( a \) to \( Q \), say \( \{a, p_4\} \), all the edges from \( b \) to \( Q \setminus \{p_4\} \) are green, and \( l = 5 \) (the edge \( \{b, p_4\} \) can be blue or green).

Let the blue \( P_1 \) in \( H \) be \( as_1s_2s_3b \). There is a vertex \( c \in H \), such that the edges \( \{a, c\} \) and \( \{b, c\} \) are green, since otherwise \( l = 6 \). This case is broken into three subcases, as follows:

1. There are at least two blue edges from \( p_4 \) to \( R \), say \( \{p_4, c_1\} \) and \( \{p_4, c_3\} \). When avoiding blue \( P_8 \), we obtain a green \( P_8 = acbp_8c_3p_7c_1p_6 \).

2. There is exactly one blue edge from \( p_4 \) to \( R \), say \( \{p_4, c_1\} \). Then, if there is at least one green edge from \( \{c_3, c_5\} \) to \( Q \setminus \{p_4\} \), say \( \{c_3, p_5\} \), then we have green \( P_8 = acbp_8c_1p_5c_3p_4 \). In the opposite case we must have a blue complete blue bipartite subgraph \( K_{\{c_1,c_3\},\{p_5,p_6,p_7,p_8\}} \). If there is at least one blue edge from \( \{p_5, p_6, p_7, p_8\} \) to \( \{s_1, s_2, s_3\} \), then we have a blue \( P_8 \), otherwise we easily find a green \( P_8 \).

3. All the edges from \( p_4 \) to \( R \) are green. Then, if there is at least one green edge from \( R \) to \( Q \setminus \{p_4\} \), say \( \{c_1, p_5\} \), then in order to avoid a green \( P_8 \), we must have a blue complete bipartite \( K_{\{c_3,c_5\},\{p_6,p_7,p_8\}} \). In the opposite case, we have a complete blue bipartite subgraph \( K_{\{c_1,c_3,c_5\},\{p_6,p_7,p_8\}} \). If there is at least one blue edge from \( \{p_6, p_7, p_8\} \) to \( \{s_1, s_2, s_3\} \), then we have a blue \( P_8 \), otherwise we have a green \( P_8 \).

\textbf{Case 4.} There is a blue edge from \( a \) to \( Q \), say \( \{a, p_4\} \), there is a blue edge from \( b \) to a different vertex in \( Q \), say \( \{b, p_8\} \), and \( l = 5 \).

Let the blue \( P_1 \) in \( H \) be \( as_1s_2s_3b \). There is a vertex \( c \in H \), such that the edges \( \{a, c\} \) and \( \{b, c\} \) are green, since otherwise \( l = 6 \). All the edges from
\{p_4, p_8\} to \( R \cup \{c\} \) are green. If there are at least two green edges from a vertex in \{p_5, p_6, p_7\} to \( R \), say \{p_5, c_1\} and \{p_5, c_3\}, then we have a green \( P_8 \), namely \( c_5p_4c_1p_5c_3p_8cb \). In the opposite case, we have a blue \( P_4 \), without loss of generality, say \( c_1p_5c_3p_6 \). To avoid a blue \( P_8 \), the edges \{a, p_6\} and \{s_1, p_6\} are green, but then we have a green \( P_8 = s_1p_6acp_4c_1p_8c_5 \). \( \square \)

It is interesting to observe that the smaller case of \( R_3(P_8) \) required significantly more complex reasoning than that of \( R_3(P_9) \). In general, we expect that even paths cases are harder than those for odd paths. Consequently, between the first two open cases of Conjecture 2, namely the questions whether it is true that \( R_3(P_{10}) = 18 \) and \( R_3(P_{11}) = 21 \), we expect the latter to be simpler to prove.

References


