Zarankiewicz Numbers and Bipartite Ramsey Numbers*

Alex F. Collins
Rochester Institute of Technology
School of Mathematical Sciences
Rochester, NY 14623
weincoll@gmail.com

Alexander W.N. Riasanovsky
University of Pennsylvania
Department of Mathematics
Philadelphia, PA 19104
alexneal@math.upenn.edu

John C. Wallace
Trinity College
Department of Mathematics
Hartford, CT 06106
john.wallace@trincoll.edu

Stanisław P. Radziszowski
Rochester Institute of Technology
Department of Computer Science
Rochester, NY 14623
spr@cs.rit.edu

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Abstract

The Zarankiewicz number $z(b; s)$ is the maximum size of a subgraph of $K_{b,b}$ which does not contain $K_{s,s}$ as a subgraph. The two-color bipartite Ramsey number $b(s, t)$ is the smallest integer b such that any coloring of the edges of $K_{b,b}$ with two colors contains a $K_{s,s}$ in the first color or a $K_{t,t}$ in the second color.

In this work, we design and exploit a computational method for bounding and computing Zarankiewicz numbers. Using it, we obtain several new values and bounds on $z(b; s)$ for $3 \le s \le 5$. Our approach and more knowledge about $z(b; s)$ permit us to improve some of the results on bipartite Ramsey numbers obtained by Goddard, Henning and Oellermann in 2000. In particular, we compute the smallest previously unknown bipartite Ramsey number, $b(2, 5) = 17$. Moreover, we prove that up to isomorphism there exists a unique 2-coloring which witnesses the lower bound $16 < b(2, 5)$. We also find tight bounds on $b(2, 2, 3)$, $17 \le b(2, 2, 3) \le 18$, which currently is the smallest open case for multicolor bipartite Ramsey numbers.

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