New Computational Upper Bounds for Ramsey Numbers $R(3, K_k - e)$

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Avoiding Triangles in Ramsey Graphs

or independence in triangle-free graphs

1. Ramsey Numbers $R(3, K_k)$ and $R(3, K_k - e)$
   - Some background and history
   - Asymptotics
   - Lower bounds on $e(3, K_k - e, n)$
   - New upper bounds on $R(3, K_k - e)$
2. New Challenges
   - Local growth of $R(3, k)$
   - Constructive lower bound on $R(3, K_k)$ and $R(3, K_k - e)$
3. So, what to do next, computationally?

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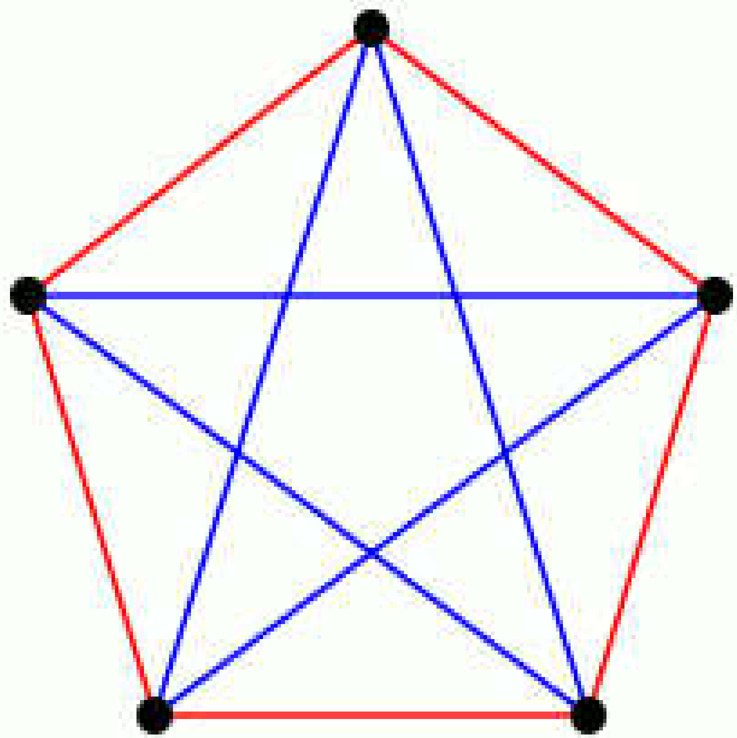
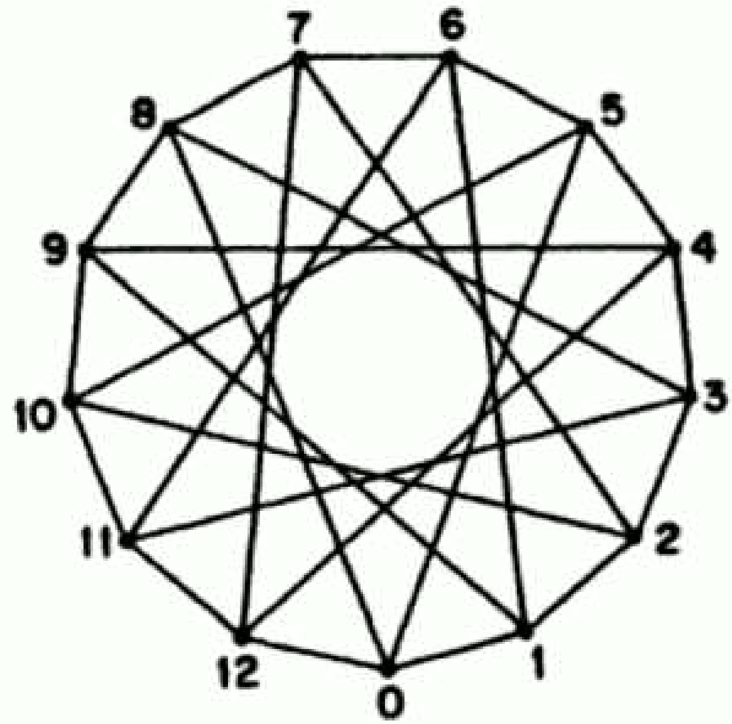
Ramsey Numbers

- $R(G, H) = n$ iff
  n = least positive integer such that in any 2-coloring of the edges of K_n there is a monochromatic G in the first color or a monochromatic H in the second color
- $R(k, l) = R(K_k, K_l)$
- generalizes to r colors, $R(G_1, \cdots, G_r)$
- *2-edge-colorings* $\cong$ *graphs*
- Theorem (Ramsey 1930): Ramsey numbers exist

Unavoidable classics



$R(3,3) = 6$



$R(3,5) = 14$ [GRS'90]

Asymptotics

diagonal Ramsey numbers

- **Bounds** (Erdős 1947, Spencer 1975, Conlon 2010)

$$\frac{\sqrt{2}}{e}2^{n/2}n < R(n,n) < R(n+1,n+1) \leq \binom{2n}{n} n^{-c\frac{\log n}{\log\log n}}$$

- **Conjecture** (Erdős 1947, \$100)

$\lim_{n\to\infty} R(n,n)^{1/n}$ exists.

If it exists, it is between $\sqrt{2}$ and 4 (\$250 for value).

Asymptotics

Ramsey graphs avoiding K_3

$$R(3, k) = \Theta\left(\frac{k^2}{\log k}\right)$$

- Kim 1995, probabilistic lower bound
- Bohman 2009, triangle-free process,
  simpler proof, more insight,
  extends to $R(4, k) = \Omega(k^{5/2}/\log k)$
- Ajtai-Komlós-Szemerédi 1980, upper bound counting edges, bounding average degree

#vertices / #triangle-free graphs

no exhaustive searches beyond 17

4	7
5	14
6	38
7	107
8	410
9	1897
10	12172
11	105071
12	1262180
13	20797002
14	467871369
15	14232552452
16	581460254001 $\approx 6 * 10^{11}$

——————too many to process——————

17	$\approx 3 * 10^{12}$

Small cases of $R(3, K_k - e)$ and $R(3, K_k)$

k	$R(3, K_k - e)$	$R(3, K_k)$	k	$R(3, K_k - e)$	$R(3, K_k)$
3	5	6	10	**37**	40–42
4	7	9	11	42–**45**	47–50
5	11	14	12	47–**53**	52–59
6	17	18	13	**55–62**	59–68
7	21	23	14	59–**71**	66–77
8	25	28	15	69–**80**	73–87
9	31	36	16	73–**91**	82–98

Ramsey numbers $R(3, K_k - e)$ and $R(3, K_k)$, for $k \leq 16$
results from this work in bold

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$e(3, K_k - e, n)$

Definition: $e(3, K_k - e, n)$ = min # edges in n-vertex triangle-free graphs G without $K_k - e$ in $\overline{G}$

- For any graph $G \in R(3, K_k - e; n, e)$

$$ne - \sum_{i=0}^{k-1} n_i(e(3, K_{k-1} - e, n - i - 1) + i^2) \geq 0$$

- Very good lower bounds on $e(3, K_{k-1} - e, n - d - 1)$ give good lower bounds on $e(3, K_k - e, n)$
- $e(3, K_k - e, n) = \infty$ implies $R(3, K_k - e) \leq n$

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K_3 versus $K_k - e$ and K_k

$$e(K_3, K_{k-1}, n) \geq e(K_3, K_k - e, n) \geq e(K_3, K_k, n)$$

$$R(K_3, K_{k-1}) \leq R(K_3, K_k - e) \leq R(K_3, K_k)$$

$\geq$ for $e()$ is much of the time $=$
$\leq$ for $R()$ seems to be close to $=$

Main computational results:

$R(K_3, K_{10} - e) = 37$
solves one of 10 open cases $R(3, G)$ for 10 vertices left by Brinkmann, Goedgebeur, Schlage-Puchta 2012

many values and bounds on $e(K_3, K_k - e, n)$

Behavior of $e(3, K_k - e, n)$

vertices	k													
n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	2													
4	4	2												
5	∞	4	2											
6		6	3	2										
7		∞	6	3	2									
8			8	4	3	2								
9			12	7	4	3	2							
10			15	10	5	4	3	2						
11			∞	14	8	5	4	3	2					
12				18	11	6	5	4	3	2				
13				24	15	9	6	5	4	3	2			
14				30	19	12	7	6	5	4	3	2		
15				35	24	15	10	7	6	5	4	3	2	
16				40	30	20	13	8	7	6	5	4	3	2
17				∞	37	25	16	11	8	7	6	5	4	3
18					43	30	20	14	9	8	7	6	5	4
19					54	37	25	17	12	9	8	7	6	5
20					60	44	30	20	15	10	9	8	7	6
21					∞	51	35	25	18	13	10	9	8	7
22						59	**42**	30	21	16	11	10	9	8
23						70	**49**	35	25	19	14	11	10	9
24						80	**56**	40	30	22	17	12	11	10
25						∞	**65**	46	35	25	20	15	12	11
26							**73**	52	40	30	23	18	13	12
27							**81**	**61**	45	35	26	21	16	13
28							**95**	**68**	51	40	30	24	19	14
29							**106**	**77**	**58**	45	35	27	22	17
30							**117**	**86**	**66**	50	40	30	25	20
31							∞	**95**	**73**	56	45	35	28	23

Exact values of $e(3, K_k - e, n)$, for $3 \le k \le 16$, $3 \le n \le 31$

$e(3, K_{k+2} - e, n)$ is known for $n < 13k/4$

Theorem. (Zhou-R 1990)
For all $n, k \geq 1$, for which $e(3, K_{k+2} - e, n)$ is finite, we have

$$e(3, K_{k+2} - e, n) = \begin{cases} 0 & \text{if } n \leq k+1, \\ n - k & \text{if } k+2 \leq n \leq 2k \text{ and } k \geq 1, \\ 3n - 5k & \text{if } 2k < n \leq 5k/2 \text{ and } k \geq 3, \\ 5n - 10k & \text{if } 5k/2 < n \leq 3k \text{ and } k \geq 6, \\ 6n - 13k & \text{if } 3k < n \leq 13k/4 - 1 \text{ and } k \geq 6. \end{cases}$$

Furthermore, $e(3, K_{k+2} - e, n) \geq 6n - 13k$ for all n and $k \geq 6$.
All critical graphs are known for $n \leq 3k$.

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Main Theorem

Theorem.
$R(3, K_{10} - e) = 37$,
$R(3, K_{11} - e) \leq 45$, $R(3, K_{12} - e) \leq 53$,
$R(3, K_{13} - e) \leq 62$, $R(3, K_{14} - e) \leq 71$,
$R(3, K_{15} - e) \leq 80$, $R(3, K_{16} - e) \leq 91$.

Proof:
$k = 10$, small $k = 11$ cases:
extenders, degree sequence analysis,
redundant computations used for consistency checks,
heavy use of McKay's *nauty*

$k \geq 12$, large $k = 11$ cases:
only degree sequence analysis,
not CPU-intensive, a few weeks of real time

$e(3, K_k - e, n), k = 11$

n	$e(K_3, K_{11} - e, n) \geq$	comments
28	51	exact
29	58	exact
30	66	exact
31	73	exact
32	80	exact, $e(3, 10, 32) = 81$
33	90	exact
34	99	exact
35	107	extender
36	117	extender
37	128	extender
38	139	extender
39	151	extender
40	161	extender
41	172	extender
42	185	$e(3, K_{10}, 42) = \infty$
43	201	
44	217	maximum 220
45	∞	hence $R(K_3, K_{11} - e) \leq 45$

Lower bounds on $e(K_3, K_{11} - e, n)$, for $n \geq 28$

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Challenge

local growth of $R(3, k)$

Erdős and Sós, 1980, asked about
$3 \leq \Delta_k = R(3, k) - R(3, k-1) \leq k$:

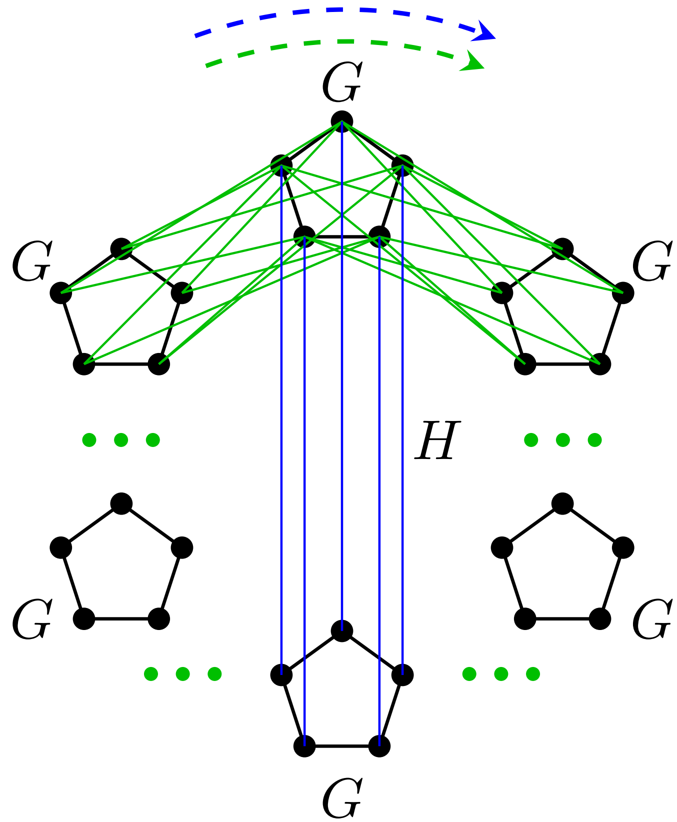
$$\Delta_k \xrightarrow{k} \infty \text{ ?} \qquad \Delta_k / k \xrightarrow{k} 0 \text{ ?}$$

Perhaps squeezing $R(3, K_k - e)$ in the middle can help.

$$\Delta_k = R(3, K_k) - R(3, K_k - e) +$$
$$R(3, K_k - e) - R(3, K_{k-1})$$

Challenge

construction by Chung/Cleve/Dagum, 1993



Construction of $H \in \mathcal{R}(3, 9; 30)$ using $G = C_5 \in \mathcal{R}(3, 3; 5)$

Challenge

constructive lower bound on $R(3, k)$

Chung/Cleve/Dagum

- start with $G \in \mathcal{R}(3, k+1; n)$
- take 6 disjoint copies of G
- this produces $H \in \mathcal{R}(3, 4k+1; 6n)$
- hence, $R(3, 4k+1) \geq 6R(3, k+1) - 5$
- $R(3, k) = \Omega(n^{\log 6/\log 4}) \approx \Omega(n^{1.29})$

Explicit $\Omega(k^{3/2})$ construction
Alon 1994, Codenotti-Pudlák-Giovanni 2000

Design a recursive construction for $R(3, k)$
better than $\Omega(k^{3/2})$

So, what to do next?

computationally

Hard but potentially feasible tasks:

Improve any of the Ramsey bounds

- $42 \leq R(3, K_{11} - e) \leq 45$
- $30 \leq R(3, 3, 4) \leq 31$
- $51 \leq R(3, 3, 3, 3) \leq 62$

Find a good lower bound on the differences

$R(3, K_k) - R(3, K_k - e)$
$R(3, K_k - e) - R(3, K_{k-1})$

Papers to pick up

- Jan Goedgebeur and Stanisław Radziszowski
  New Computational Upper Bounds for Ramsey Numbers
  $R(3, k)$, *EIJC*, 20(1) (2013) #P30, 28 pages.

- SPR's survey *Small Ramsey Numbers* at the *EIJC*
  Dynamic Survey DS1, revision #13, August 2011
  `http://www.combinatorics.org`

  All references therein

Thanks
for listening