New Computational Upper Bounds for Ramsey Numbers $R(3,k)$

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Avoiding Triangles in Ramsey Graphs

or independence in triangle-free graphs

1. Ramsey Numbers $R(3,k)$
   - Asymptotics
   - Some background and history
   - Lower bounds on $e(3,k,n)$
   - Upper bounds on $R(3,k)$
2. New Challenges
   - Local growth of $R(3,k)$
   - Constructive lower bound on $R(3,k)$
   - Work more on $R(3,K_k-e)$
3. So, what to do next, computationally?

Ramsey Numbers

- $R(G, H) = n$ iff
  n = least positive integer such that in any 2-coloring of the edges of K_n there is a monochromatic G in the first color or a monochromatic H in the second color
- $R(k, l) = R(K_k, K_l)$
- generalizes to r colors, $R(G_1, \cdots, G_r)$
- 2-*edge-colorings* $\cong$ *graphs*
- Theorem (Ramsey 1930): Ramsey numbers exist

Asymptotics

diagonal Ramsey numbers

- **Bounds** (Erdős 1947, Spencer 1975, Conlon 2010)

$$\frac{\sqrt{2}}{e}2^{n/2}n < R(n,n) < R(n+1,n+1) \leq \binom{2n}{n} n^{-c\frac{\log n}{\log\log n}}$$

- **Conjecture** (Erdős 1947, \$100)
  $\lim_{n\to\infty} R(n,n)^{1/n}$ exists.

  If it exists, it is between $\sqrt{2}$ and 4 (\$250 for value).

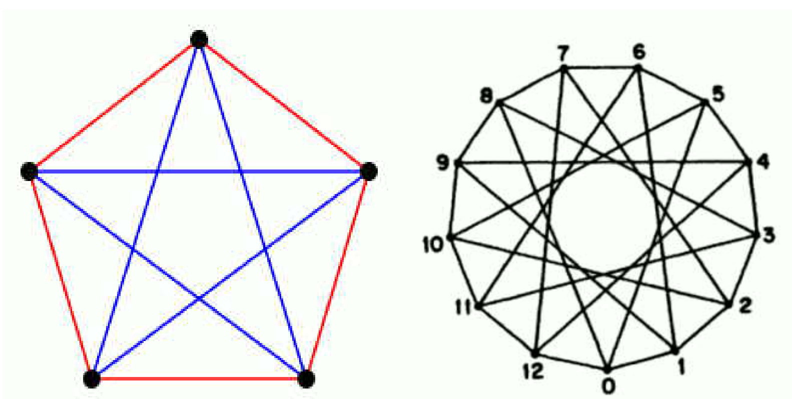
Asymptotics

Ramsey graphs avoiding K_3

$$R(3,k) = \Theta\left(\frac{k^2}{\log k}\right)$$

- Kim 1995, probabilistic lower bound
- Bohman 2009, triangle-free process, simpler proof, more insight, extends to $R(4,k) = \Omega(k^{5/2}/\log k)$
- Ajtai-Komlós-Szemerédi 1980, upper bound counting edges, bounding average degree

Unavoidable classics



$R(3,3) = 6$ $R(3,5) = 14$ [GRS'90]

#vertices / #triangle-free graphs

no exhaustive searches beyond 17

4	7
5	14
6	38
7	107
8	410
9	1897
10	12172
11	105071
12	1262180
13	20797002
14	467871369
15	14232552452
16	581460254001 $\approx 6 * 10^{11}$

——————too many to process——————

17	$\approx 3 * 10^{12}$

Values and bounds on $R(k, l)$

two colors, avoiding K_k, K_l

k \ l	3	4	5	6	7	8	9	10	11	12	13	14	15
3	6	9	14	18	23	28	36	40 42	46 50	52 59	59 68	66 77	73 87
4		18	25	36 41	49 61	56 84	73 115	92 149	98 191	128 238	133 291	141 349	153 417
5			43 49	58 87	80 143	101 216	126 316	144 442	171 633	191 848	213 1139	239 1461	265 1878
6				102 165	113 298	132 495	169 780	179 1171	253 1804	263 2566	317 3705	5033	401 6911
7					205 540	217 1031	241 1713	289 2826	405 4553	417 6954	511 10581	15263	22116
8						282 1870	317 3583	6090	10630	16944	817 27490	41525	861 63620
9							565 6588	581 12677	22325	39025	64871	89203	
10								798 23556		81200			1265

[ElJC survey *Small Ramsey Numbers*, revision #13, 2011, with 2012 updates]

$R(3, k)$

new upper bounds in bold

k	$R(3,k)$	k	$R(3,k)$
3	6	10	40–**42**
4	9	11	46–**50**
5	14	12	52–59
6	18	13	59–**68**
7	23	14	66–**77**
8	28	15	73–**87**
9	36	16	79–**98***

Values and bounds on Ramsey numbers $R(3, k)$, for $k \leq 16$

Small $R(3, k)$ cases

k	$R(3,k)$	year	reference [lower/upper]
3	6	1953	Putnam Competition
4	9	1955	Greenwood-Gleason
5	14	1955	Greenwood-Gleason
6	18	1964	Kéry
7	23	1966 / 1968	Kalbfleisch / Graver-Yackel
8	28	1982 / 1992	Grinstead-Roberts / McKay-Zhang
9	36	1966 / 1982	Kalbfleisch / Grinstead-Roberts
10	40-43	1989 / 1988	Exoo / Kreher-R

Known values of $R(3, k)$

$e(3, k, n)$

Definition: $e(3, k, n)$ = min # edges in n-vertex triangle-free graphs without independent sets of order k

- Very good lower bounds on $e(3, k-1, n-d)$ give good lower bounds on $e(3, k, n)$
- For any graph $G \in R(3, k; n, e)$

$$ne - \sum_{i=0}^{k-1} n_i(e(3, k-1, n-i-1) + i^2) \geq 0$$

- $e(3, k, n) = \infty$ implies $R(3, k) \leq n$

$e(3, k+1, n)$ is known for $n < 13k/4$

Theorem. (Kreher-R – 1988, 1991)
For all $n, k \geq 1$, for which $e(3, k+1, n)$ is finite,

$$e(3, k+1, n) = \begin{cases} 0 & \text{if } n \leq k, \\ n-k & \text{if } k < n \leq 2k, \\ 3n-5k & \text{if } 2k < n \leq 5k/2, \\ 5n-10k & \text{if } 5k/2 < n \leq 3k, \\ 6n-13k & \text{if } 3k < n \leq 13k/4 - 1. \end{cases}$$

Furthermore, $e(3, k+1, n) \geq 6n - 13k$ holds for all n and k.
All critical graphs are known for $n \leq 3k$.

Behavior of $e(3, k, n)$

vertices n	k = 3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	1													
4	2	1												
5	5	2	1											
6	∞	3	2	1										
7		6	3	2	1									
8		10	4	3	2	1								
9		∞	7	4	3	2	1							
10			10	5	4	3	2	1						
11			15	8	5	4	3	2	1					
12			20	11	6	5	4	3	2	1				
13			26	15	9	6	5	4	3	2	1			
14			∞	20	12	7	6	5	4	3	2	1		
15				25	15	10	7	6	5	4	3	2	1	
16				**32**	20	13	8	7	6	5	4	3	2	1
17				**40**	25	16	11	8	7	6	5	4	3	2
18				∞	30	20	14	9	8	7	6	5	4	3
19					**37**	25	17	12	9	8	7	6	5	4
20					**44**	30	20	15	10	9	8	7	6	5
21					**51**	35	25	18	13	10	9	8	7	6
22					**60**	**42**	30	21	16	11	10	9	8	7
23					∞	**49**	35	25	19	14	11	10	9	8
24						**56**	40	30	22	17	12	11	10	9
25						**65**	46	35	25	20	15	12	11	10
26						**73**	52	40	30	23	18	13	12	11
27						**85**	**61**	45	35	26	21	16	13	12
28						∞	**68**	51	40	30	24	19	14	13
29							**77**	**58**	45	35	27	22	17	14
30							**86**	**66**	50	40	30	25	20	15
31							**95**	**73**	56	45	35	28	23	18

Exact values of $e(3, k, n)$, for $3 \leq k \leq 16$, $3 \leq n \leq 31$.

Behavior of $e(3, k+1, n)$ for $n > 13k/4$

mainly computations

Two types of computations:
extension algorithms and degree sequence analysis

- Grinstead-Roberts, 1982
- McKay-Zhang, 1992
- Kreher-R, 1988, 1991
- Lesser, 2001
- Backelin, 2001-2012
- This work:
  all exact values $e(3, \leq 9, n)$
  exact values $e(3, 10, \leq 34)$
  many new lower bounds on $e(3, \geq 10, n)$

Main results on $e(3, k, n)$

based on computations and reasonings

Theorem.
There exists a unique $(3, 9; 35)$-graph,
and $e(3, 9, 35) = 140$.

[cyclic $(3, 9; 35)$-graph, *dist* ={1, 7, 11, 16}, Kalbfleisch, 1966]

Theorem.
$R(3, 10) = 43$ if and only if $e(3, 10, 42) = 189$.

[Any $(3, 10; 42, 189)$-graph would be 9-regular]

Main Theorem

Theorem.
$R(3, 10) \leq 42$, $R(3, 11) \leq 50$,
$R(3, 13) \leq 68$, $R(3, 14) \leq 77$,
$R(3, 15) \leq 87$, $R(3, 16) \leq 98^*$.

Proof:
$k = 10$: extenders, tricks, degree sequence analysis
about 50 CPU years, heavy use of McKay's *nauty*
redundant computations used for consistency checks

$k \geq 11$: only degree sequence analysis
not CPU-intensive, a few months of real time

$(3, 10)$-graphs

$R(3, 10) \leq 42$

- Any $(3, 10; 42, 189)$-graph is an extension of a $(3, 9; 32, 108)$-graph
- There are 2104151 $(3, 9; 32, 108)$-graphs, these were obtained with a significant effort
- None of them extends to a 9-regular $(3, 10; 42, 189)$-graph

$R(3, 10) \geq 40$

- Exoo found 266K+ $(3, 10; 39)$-graphs
- We found 37M+ $(3, 10; 39)$-graphs G, $161 \leq e(G) \leq 175$ none of them extends to a $(3, 10; 40)$-graph

$e(3, k, n), k = 11$

n	$e(3, 11, n) \geq$	comments
32	62t	63 - Lesser, 63 is exact - Backelin
33	68t	69 - Lesser, 70 is exact - Backelin
34	75	76 - Lesser, 77 - Backelin
35	83	84 in Lesser/Backelin
36	92	
37	100	
38	109	
39	117	unique solution, 6-regular
40	128	
41	138	
42	149	
43	159	
44	170	
45	182	
46	195	199 required for $R(3, 12) \leq 58$
47	209	
48	222	unique solution: $n_9 = 36$, $n_{10} = 12$, 215 required for $R(3, 12) \leq 59$, old bound
49	237	245 maximum
50	∞	hence $R(3, 11) \leq 50$, new bound
51	∞	hence $R(3, 11) \leq 51$, old bound

Lower bounds on $e(3, 11, n)$, for $n \geq 32$.

Challenge

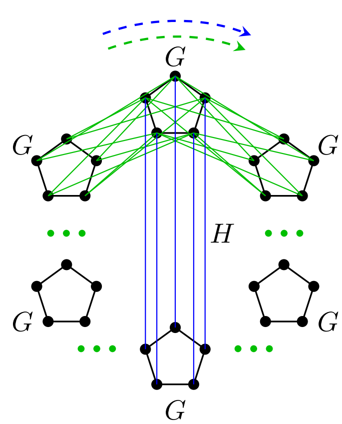
local growth of $R(3, k)$

Erdős and Sós, 1980, asked about
$3 \leq \Delta_k = R(3, k) - R(3, k-1) \leq k$:

$$\Delta_k \xrightarrow{k} \infty \, ? \qquad \Delta_k / k \xrightarrow{k} 0 \, ?$$

Challenge

construction by Chung/Cleve/Dagum, 1993



Construction of $H \in \mathcal{R}(3, 9; 30)$ using $G = C_5 \in \mathcal{R}(3, 3; 5)$

Challenge

constructive lower bound on $R(3, k)$

Chung/Cleve/Dagum

- start with $G \in \mathcal{R}(3, k+1; n)$
- take 6 disjoint copies of G
- this produces $H \in \mathcal{R}(3, 4k+1; 6n)$
- hence, $R(3, 4k+1) \geq 6R(3, k+1) - 5$
- $R(3, k) = \Omega(n^{\log 6/\log 4}) \approx \Omega(n^{1.29})$

Explicit $\Omega(k^{3/2})$ construction
Alon 1994, Codenotti-Pudlák-Giovanni 2000

Design a recursive construction for $R(3, k)$
better than $\Omega(k^{3/2})$

Challenge

K_3 versus $K_k - e$

$R(3, K_7 - e) = 21$	$R(3, K_8 - e) = 25$	$R(3, K_9 - e) = 31$
$R(3, 7) = 23$	$R(3, 8) = 28$	$R(3, 9) = 36$

All $R(3, K_k - e)$ critical graphs are known for $k \leq 8$
All $R(3, K_k)$ critical graphs are known for $k \leq 9$

First open cases:

$37 \leq R(K_3, K_{10} - e) \leq 38$, $42 \leq R(K_3, K_{11} - e) \leq 47$
$40 \leq R(K_3, K_{10}) \leq 42$, $46 \leq R(K_3, K_{11}) \leq 50$

So, what to do next?

computationally

Hard but potentially feasible tasks:
Improve any of the Ramsey bounds

- $37 \leq R(3, K_{10} - e) \leq 38$
- $42 \leq R(K_3, K_{11} - e) \leq 47$
- $30 \leq R(3, 3, 4) \leq 31$
- $51 \leq R(3, 3, 3, 3) \leq 62$
- Find a good lower bound on the difference $R(3, K_k) - R(3, K_k - e)$.

Papers to pick up

- Jan Goedgebeur and Stanisław Radziszowski
  New Computational Upper Bounds for Ramsey Numbers $R(3, k)$, arXiv eprint (2012)

- SPR's survey *Small Ramsey Numbers* at the *EIJC* Dynamic Survey DS1, revision #13, August 2011
  `http://www.combinatorics.org`

  All references therein

Thanks
for listening