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Convolutions Combinatorially

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Abstract

Inspired by Benjamin and Quinn's combinatorial approach to the Fibonacci numbers, etc. (*Proofs that Really Count, The Art of Combinatorial Proof*), we observe that the Fibonacci convolution sequence has a conceptually simple combinatorial interpretation which leads immediately to a fourth-order recurrence and other properties.

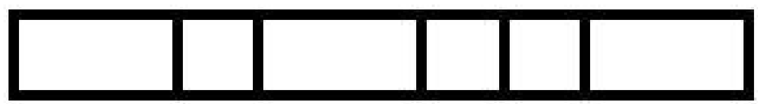
We also find a combinatorial nearly proof-without-words for the convolution formula for the difference between the Pell and Fibonacci sequences as well as differences for other linear recurrences. This uses the observation that the number of ways to tile an n-board with squares and dominoes is a Fibonacci number; in case the squares come in two colors, it's a Pell number.

Finally, a consideration of the placement of the second color in tiling an n-board gives the Pell numbers as another convolution of the Fibonacci numbers.

Fibonacci and Lucas Numbers

How many ways can you tile an n-board with squares and dominoes?

Answer: the Fibonacci number f_n



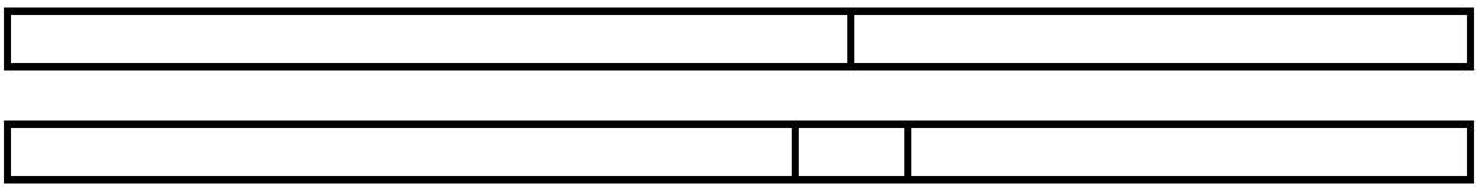
How many ways can you tile an n-bracelet with 1-beads and 2-beads?

Answer: the Lucas number L_n

Why B & Q's Approach is Good

Tile an $(a+b)$–board f_{a+b} ways with squares and dominoes.

Cover positions a and $a+1$ with a domino, or don't.



$$f_{a+b} = f_a f_b + f_{a-1} f_{b-1}$$

All the Tilings of an n-Board

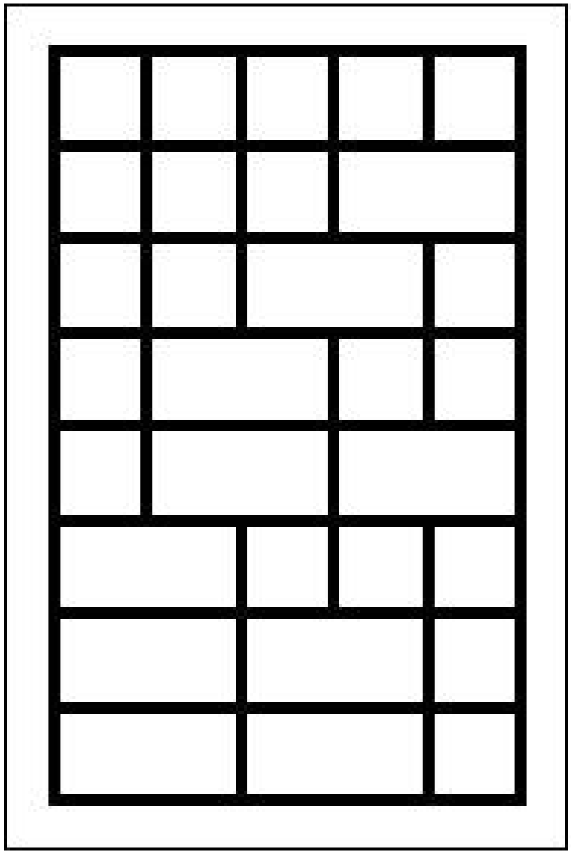
$n = 5$, $f_n = 8$.

All $f_5 = 8$ tilings of a 5-board.

We use $A_5 = 30$ tiles in all.

$A'_5 = 20$ squares

$A''_5 = 10$ dominoes



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A Recurrence for A_n

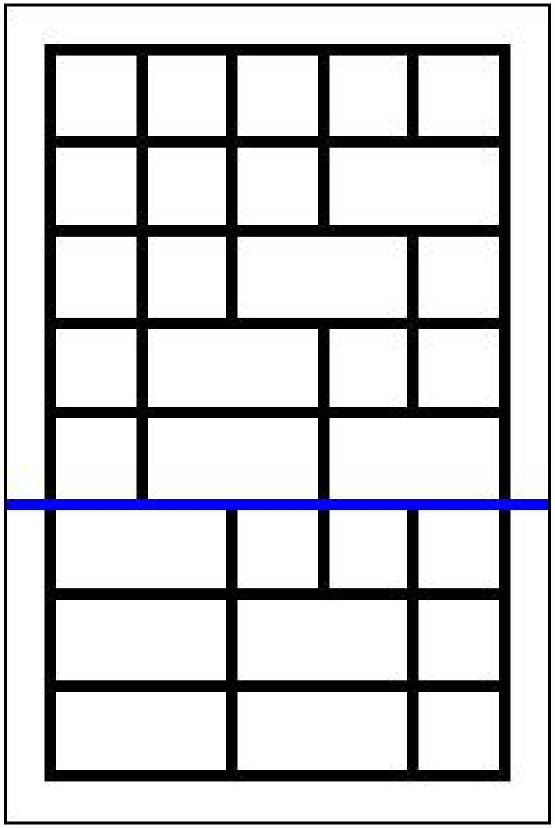
f_{n-1} rows begin with a square with A_{n-1} other tiles

f_{n-2} rows begin with a domino with A_{n-2} other tiles

This gives the recurrence

$$A_0 = 0, \quad A_1 = 1$$

$$A_n = A_{n-1} + A_{n-2} + f_n$$



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All Three Recurrences (from the Picture)

$$A'_0 = 0, \quad A'_1 = 1, \quad A'_n = A'_{n-1} + A'_{n-2} + f_{n-1}$$

$$A''_0 = 0, \quad A''_1 = 0, \quad A''_n = A''_{n-1} + A''_{n-2} + f_{n-2}$$

$$A_0 = 0, \quad A_1 = 1, \quad A_n = A_{n-1} + A_{n-2} + f_n$$

$\{A'_n\}$ and $\{A''_n\}$ are Sloane's A001629

A001629 is the Fibonacci convolution sequence $(F * F)_n = \sum_0^n F_i F_{n-i}$.

The first few terms of A' are 0, 1, 2, 5, 10, 20, 38, 71, 130, 235, 420

The first few terms of A'' are 0, 0, 1, 2, 5, 10, 20, 38, 71, 130, 235

The first few terms of A are 0, 1, 3, 7, 15, 30, 58, 109, 201, 365, 655 (unlisted numbers)

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Counting Beads: There are L_n n-Bracelets

In all L_n n-bracelets: B'_n 1-beads, B''_n 2-beads, & $B_n = B'_n + B''_n$ total.

The first few values of B' are: 1, 2, 6, 12, 25, 48, 91, 168, 306, 550 (unlisted)

The first few values of B'' are: 0, 2, 3, 8, 15, 30, 56, 104, 104, 189, 340 (unlisted)

The first few values of B are: 1, 4, 9, 20, 40, 78, 147, 272, 495, 890 (A0023607)

$$B_n = nF_n = \sum_{0}^{n} L_i F_{n-i}$$

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All Three Recurrences

$$B'_1 = 1, \quad B'_2 = 2, \quad B'_n = B'_{n-1} + B'_{n-2} + L_{n-1}$$

$$B''_1 = 0, \quad B''_2 = 2, \quad B''_n = B''_{n-1} + B''_{n-2} + L_{n-2}$$

$$B_1 = 1, \quad B_2 = 4, \quad B_n = B_{n-1} + B_{n-2} + L_n$$

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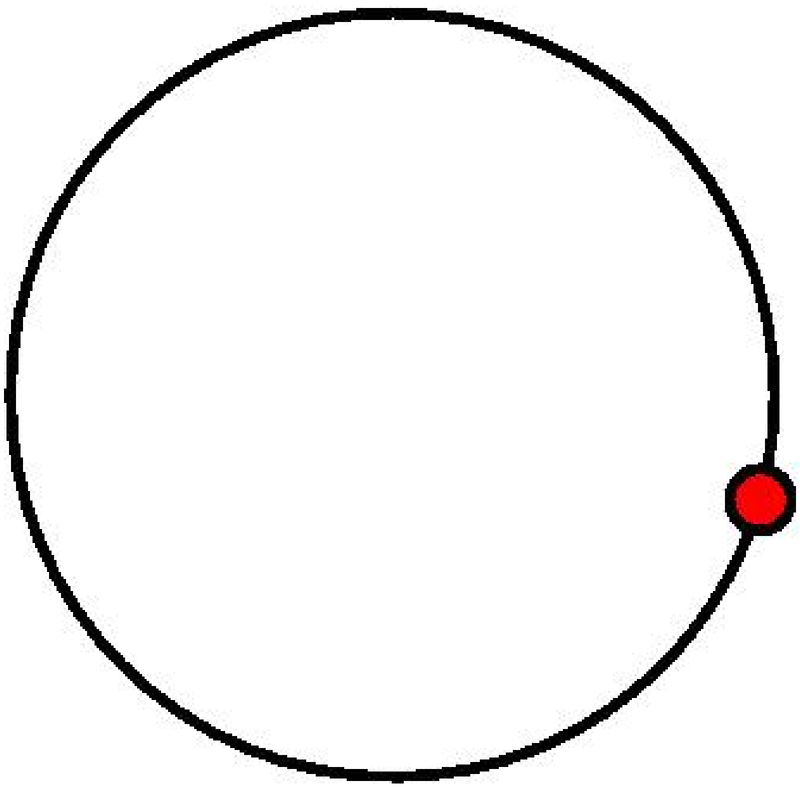
Tile Bracelet With One Red 1-Bead



With white 1-beads & 2-beads and *one* red 1-bead, tiling count is m_n:

$$m_1 = 1,\ m_2 = 2,\ m_n = m_{n-1} + m_{n-2} + L_{n-1}$$

Proof: Remove the first bead after the bead covering position 1.

Clearly, $m_n = nf_{n-1}$

No more begging the question:

$$m_n = B'_n$$

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Tile With One Red Square

With white squares & dominoes and *one* red square, tiling count is g_n:

$$g_0 = 0, \quad g_1 = 1, \quad g_n = g_{n-1} + g_{n-2} + f_{n-1}$$

Proof: Remove the first tile.

If tile k is red, the prefix $k-1$ and suffix $n-k$ boards are Fibonacci-tiled.

$$g_n = \sum_{k=1}^{n} f_{k-1} f_{n-k}$$

No more begging the question: $g_n = A'_n$.

More Varied Tiles: Sizes and Colors

With red & white squares and white dominoes, the tiling count is $p_n = P_{n+1}$ (P is the Pell sequence): $p_0 = 1, \;\; p_1 = 2, \;\; p_n = 2p_{n-1} + p_{n-2}$.

If the first red square (if any) is in position k, the prefix $(k-1)$–board is tiled f_{k-1} ways, and the suffix $(n-k)$–board is tiled p_{n-k} ways.

$$p_n - f_n = \sum_{k=1}^{n} f_{k-1} p_{n-k}$$

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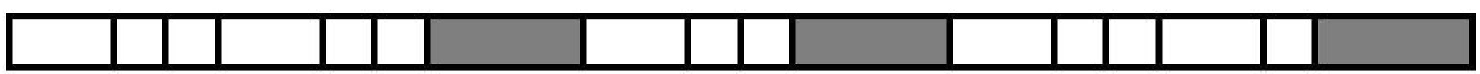
Tribonacci Numbers

The Tribonacci number t_n counts the tilings of an n-board with squares, dominoes, and trimonoes

Left of the first trimono (if any) is a Fibonacci tiling.

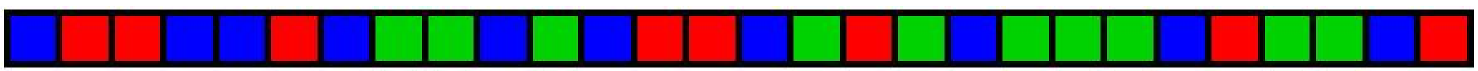
$$t_n - f_n = \sum_{k=1}^{n} f_{k-1} t_{n-k-2}$$



Tiling with Red, Blue, and Green Squares

Where is the first Green square?

$$3^n - 2^n = \sum_{k=1}^{n} 2^{k-1} 3^{n-k}$$



Convolutions on Steroids

p_n counts the tilings of an n-board with dominoes and red & white squares.

A tiling with k red squares has $k+1$ gaps filled with white squares and dominoes tiled in a Fibonacci number of ways.

$$p_n = \sum_{k=0}^{n} \sum_{a_1+\cdots+a_{k+1}=n-k} \prod_{i=1}^{k+1} f_{a_i}$$

Conclusion

The combinatorial approach is good.

Rounding out the last few identities to bracelet form is left to the interested listener....