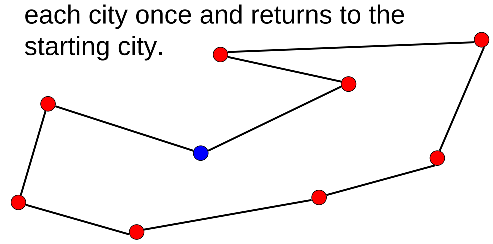
NP-Completeness

1. Polynomial time algorithm
2. Polynomial time reduction
3. P vs NP
4. NP-completeness

(some slides by P.T. Uma University of Texas at Dallas are used)

Traveling Salesperson Problem

- Find minimum length tour that visits each city once and returns to the starting city.



Given a Problem

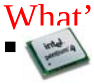
- Polynomial time algorithm ☺
  - $O(n^k)$ (n is the input size; k is a constant)
- Super-polynomial time algorithm ☹

(some people call it non-polynomial)
  - $O(2^n)$, $O(n!)$

What's the Big Deal?

-  2.20 GHz (Jan 2002)
  (2.20 billion cycles per second)
- N = 30 cities N! = 30! = 265 * 10^{30} tours
- Super-fast machine: compute 1 TSP tour in 1 cycle.
- 2.20 billion TSP tours in 1 second.
- 120 * 10^{21} seconds
- 1 year = 31536000 s
- 38 * 10^{14} years!
- 3800 trillion years!!!
- Age of Earth = 4.6 billion years!

What's the Big Deal?

-  2.53 GHz (Aug 2002)
  (2.53 billion cycles per second)
- N = 30 cities N! = 30! = 265 * 10^{30} tours
- Super-fast machine: compute 1 TSP tour in 1 cycle.
- 2.53 billion TSP tours in 1 second.
- 104 * 10^{21} seconds
- 1 year = 31536000 s
- 32 * 10^{14} years!
- 3200 trillion years!!!
- Age of Earth = 4.6 billion years!

What's the Big Deal?

-  3.06 GHz (Jan 2003)
  (3.06 billion cycles per second)
- N = 30 cities N! = 30! = 265 * 10^{30} tours
- Super-fast machine: compute 1 TSP tour in 1 cycle.
- 3.06 billion TSP tours in 1 second.
- 86 * 10^{21} seconds
- 1 year = 31536000 s
- 27 * 10^{14} years!
- 2700 trillion years!!!
- Age of Earth = 4.6 billion years!

Progress in Technology

- N = 30 cities N! = 30! = 265 * 10^{30} tours
- Super-fast machine: compute 1 TSP tour in 1 cycle.
- Age of Earth (earth) = 4.6 billion years.
- Age of the Milky Way Galaxy (mwg) = 13 billion years.
- Age of the Universe (univ) = 15 billion years.

Jan 2002	Aug 2002	Jan 2003
3800 trillion years	3200 trillion years	2700 trillion years
826, 086 earth	695, 652 earth	586, 956 earth
292, 307 mwg	246, 153 mwg	207, 692 mwg
253, 333 univ	213, 333 univ	180, 000 univ

Given a Problem

- Tractable or intractable?
- Tractable – give a polynomial time algorithm.
- Intractable – show the problem is NP-complete and explore other means of solving the problem.

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- Tractable or intractable?
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Given a Problem

- ***Give an efficient polynomial time algorithm.***
- 3 GHz ; 3 billion cycles/s ; 0.33 ns/cycle
- N = 1, 000, 000
- O(n) = 330 µs
- O(n^2) = 330 s = 5.5 minutes
- O(n^3) = 330 million s = 10 years

1. NP-Completeness

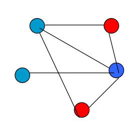
Before defining *P* and *NP*, let's understand the differences between problem that require to

1) give a YES or NO answer (***decision problem***)
2) find the cost of the optimal solution
3) find the optimal solution.

for example
1) Does the graph contain a spanning tree with weight at most 40?
2) What is the weight of the minimum spanning tree.
3) Find the minimum spanning tree

More Example: <u>colouring</u>

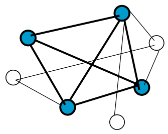
Given a graph *G*=<*V*,*E*>, we want to paint the vertices so that for any (*v*,*u*) in *E*, the colour of *v* is different from the colour of *u*.



1) **COLD:** On input *G* and an integer *k*, is *G* is *k*-colourable?
(a special case when *k*=3, is know as the 3-colour problem, **3COL**).
2) **COLO:** On input *G* what is the minimum number of colours required to paint the graph?
3) **COLC:** On input *G*, find the way to paint the graph with minimum colours.

More Example: *k*-clique

Given a graph $G=<V,E>$, we want to find a subgraph of G that is a complete graph. A graph is complete if any two vertices are connected by an edge. A complete graph with k vertices is also known as a k-clique.



this graph contains a 4-clique

1) **CLQD:** On input G and an integer k, determine whether G contains a k-clique.
2) **CLQO:** On input G find the size of the largest clique.
3) **CLQC:** Find the largest clique in G.

1. Polynomial time algorithm

A algorithm is polynomial time if its worst-case running time is in $O(n^k)$ where n is the size of the input, and k is a constant independent of n.

For example,

quicksort is polynomial time $O(n^2)$.
mergesort is polynomial time $O(n \log n)$ which is also in $O(n^2)$.
Prim's algorithm is polynomial time $O(|V| \log |V| + |E|)$.
 if we take the $(n=|V|+|E|)$ as the size of the input, then Prim's algorithm is in $O(n \log n)$.

The following is not a polynomial time algorihtm.

1) input an n-bit integer M.
2) for i=1 to M; print i; end;

Note that the size of the problem is n.
The running time is $O(2^n)$, which is not a polynomial.

2. Polynomial time reduction

Suppose we have an algorithm, known as the *oracle*, that can determine whether a graph has a k-clique in O(1) worst case running time, can we find the k-clique easily?

In other words, if we can solve the decision problem, can we solve the other 2 forms of problem?

CLQO: On input G find the size of the largest clique.

To find the size of the largest clique, we can ask the oracle in the following way,

For i=n down to 1
 If the graph contains a i-clique, return (i).
end

The worst case running time is O(n), which is a polynomial.
(the running time can be improved to O(log n)).

Suppose we can solve **CLQO** in O(1) time, can we solve **CLQC** efficiently?

3) **CLQC:** Find the largest clique in G.

1. Let T be the set of all vertices. Let G' be the graph G.
2. Ask the oracle the size of the largest clique in G. Let k be the size.
2. Select a vertex v from T. Remove v and all edges incident to v from G'.
3. Ask the oracle about G'. Let k' be the size of the largest clique in G'.
4. If k not equal k', then put v and all edges remove in step 2 back to G'.
5. Repeat step 2 to step 4 until T is empty.
6. Output G'.

The running time of the above is $O(n^2)$ where n is the number of vertices in G.

Definition:
Let A and B be two problems.
We say that A is polynomially Turing reducible to B if
 there exists a polynomial time algorithm for solving A if we could solve B in O(1) time.

If A is polynomially Turing reducible to B, we write
 $A \le B$ (or $B \ge A$)

If $A \le B$ and $B \le A$, we say that A and B are polynomially Turing equivalent, written as $A = B$.

If $A \le B$, we can view "B is more difficult or equal to A, because if we can solve B, then we can solve A".

Properties of reduction:
1) If $A \geq B$ and $B \geq C$, then $A \geq C$.
2) $A \geq A$.

Recall that **CLQD** $\geq$ **CLQO** and **CLQO** $\geq$ **CLQC**
Furthermore, it is clear that **CLQC** $\geq$ **CLQD.**

Thus we have **CLQD = CLQO = CLQC.**

So, the 3 problems are actually equivalent.

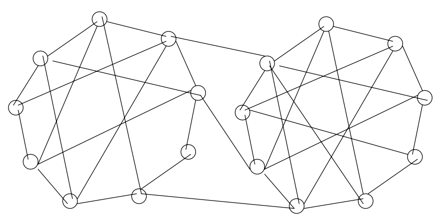
Tutorial:
Show that COLD = COLO = COLC.

Remark

This lecture note taking "short-cut" in defining polynomial Turing reducible. The notation used for *polynomial Turing reducible* is usually this : $\geq_T$,
which is to be distinguish from *polynomial many-to-one reducible*, usually denote as: $\geq_m$.

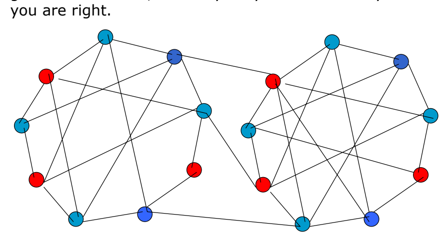
For polynomial many-to-one reducible, we can only call the oracle once. In polynomail Turing reducible, we can use it polynomial number of times.

If you tell me that this graph is 3-colourable,



it is very difficult for me to check whether you are right.

But if you tell me that this graph is 3-colorable and give me a solution, it is very easy for me to verify whether you are right.



Loosely speaking, problems that are difficult to compute, but easy to verify are known as Non-deterministic Polynomial.

P vs NP

Definition: P
P is the set of decision problems that can be solved by a polynomial time algorithm.

Recall that an algorithm is polynomial time if its worst-case running time is in $O(n^k)$ where n is the size of the input, and *k* is a constant independent of *n*.

We can represent a decision problem using a set, say K. An instance *x* is in K iff on input *x*, the output is YES.

For example, let K_1 be the problem where given an input, output YES if the input is already sorted in increasing order.
Then, K_1 is the set of sequences which are sorted,
$K_1 = \{\langle\rangle , \langle 1,2\rangle, \langle 1,3\rangle , \langle 2,3\rangle, \langle 1,2,3\rangle, \langle 1,4\rangle .... \}$

It is easy to write a linear time algorithm for K_1, thus, K_1 is in P (or we simply write $K_1 \in P$).

More examples:

1. Let K_2 be the set of binary sequence whose binary representation is dividable by 3.
$K_2 = \{ 11, 110, 1001, 1100, 1111,.....\} = \{3,6,9,12,15,...\}$
K_2 is in P.
(the length of the input "15" is 4, because $15 = 1111_2$)

2. Let K_3 be the set of binary sequence whose binary representation is a prime.
$K_3 = \{10,11,101, 111, 1011, 1101,...\} = \{2,3,5,7,11,13,...\}$

For many hundreds of years, we don't have an algorithm that can solve K_3 in polynomial time, although people believe that there should be one. Recently, researchers from India find a polynomial time algorithm, i.e. they prove that $K_3 \in P$.

3. Let K_4 be the set of weighted graphs whose Minimum Spanning Tree have weight less than 30. Then $K_4 \in P$.

Definition: NP (non-deterministic polynomial)
A decision problem K is NP iff, there exists a Q∈P s.t.
x∈K if and only if there exists a y s.t. ⟨x,y⟩∈Q.

For example, let Q be the set of <x,y>, where x is dividable by y , where (y>1) and (x>y). Here x and y are represented as binary sequences.
Q={ <100,10>,<110,10>,...... }
={ <4,2>,<6,2>,<6,3>,<8,2>,<8,4>,<9,3>,<10,2>,<10,5>,..}
Note that Q∈P.

Let K be the set of binary sequences, which represent a non-prime number that is greater than 1. (For many years, no one know whether K∈P. Recently, researchers from India prove that K∈P).
By the above definition, clearly, K∈NP.
This is because a number x is non-prime iff there exists a y>1 and x>y s.t. x is dividisible by y.

For eg., 135 is not a prime because <135,5>∈Q.
13 is a prime because there don't exists any y s.t. <13,y>∈Q.

The y in the definition is known as the *witness* or *certificate* or *proof* that x∈K.

The problem *Q* is known as the *proof system*.

So, non-deterministic polynomial are problems that have a proof system that can be solved in polynomial time.

In other words, non-deterministic polynomial are problems that can be easily verified in polynomial time.

We say that a decision problem is *decision-reducible* if, given an oracle that solves the decision problem in O(1) time, we can find the witness in polynomial time.

More examples of NP problems.

1. 3COL (3-colorability) is in NP.

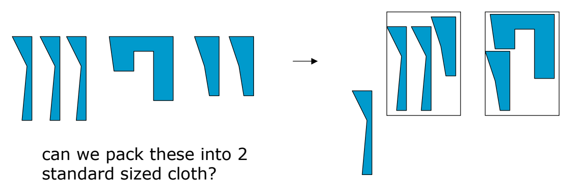
2. CLQD (k-clique) is in NP.

3. Given a sequence of integer, $a_1,a_2,a_3,....,a_n$, and an integer *k*, can we group them into *k* groups s.t. the sum of each group is less than 50.

4. **Partition problem**. Given a sequence of integers, $a_1,a_2,a_3,....,a_n$, can we group them into 2 groups s.t. the sum of each group is the same.

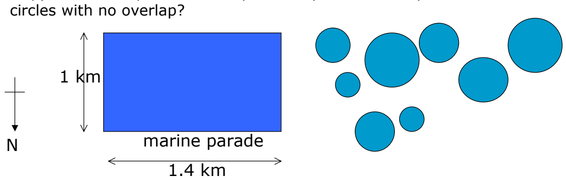
5. **Packing**: Given a set of template for the *n* parts in a jean, and *k* pieces of standard sized cloth.
Can we cut them out from *k* pieces of standard sized cloth.

In the optimization version, we want to know how to cut them from minimum number of standard sized cloth.



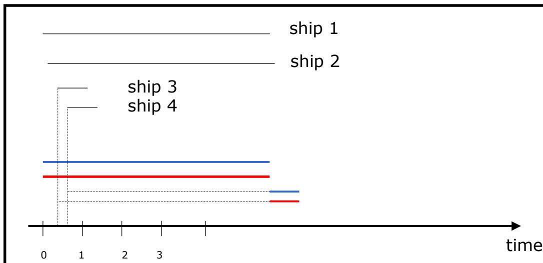
can we pack these into 2 standard sized cloth?

6. Ship **parking**..
We have *n* ships, where each ship is represented as a circle with certain radius. (the radius depend on the size of the ship). Could we park these ships in the port? That is, pack the circles with no overlap?



7. **Scheduling**.

Suppose our harbor has 2 docking facilities A and B. Thus we can serve 2 ship concurrently.
Given a list of *n* ships, and the expected docking time and arrival time of each ship. Can we assign the ship to either A or B so that each ship will not wait for more than 3 hours, and the average waiting time is less than 1 hour?



8. Scheduling. Given the modules taken by *n* students, can we plan a two-week exam so that there is no crash, and each student will take at most two exams in any 3 consecutive days.

Hundreds of "NP-hard" problems are listed in
M.R.Garey, D.S. Johnson, *Computers and Intractability: a guide to the theory of NP-Completeness*, 1979, Freeman.

So, NP are important problems.

Theorem $P \subseteq NP$

This theorem states that any problem that can be solved in polynomial time, can also be verified in polynomial time.

(this is so obviously true....)
In proof, let K be a problem in P. Let us consider this problem Q which is defined as
$Q = \{ <x,0> \mid x \in K \}$.
Now, Q can be a proof system for K, and thus $K \in NP$.
Since for any $K \in P$, we have $K \in NP$, therefore $P \subseteq NP$.

Now, the million dollars open problem is,

is $NP \subseteq P$?

if this is true (i.e, NP=P) , then any problem in NP can be solved efficiently.
A lot of researchers have worked on some NP problems but get no progress.

So, there might be some problems in NP that cannot be solved efficiently. (i.e. $NP \neq P$).

Unfortunately, we still don't know the answer. Most people strongly believe that $NP \neq P$.

4. NP-complete

Definition: NP-hard
A decision problem K is NP-hard if
1) $K \geq Y$ for every $Y \in NP$.

Definition: NP-complete
A decision problem K is NP-complete if
1) $K \in NP$, and
2) K is NP-hard.

The first definition can be viewed as: K is more difficult or equal to any problem in NP.

Note that a NP-complete problem K is the "ticket" to all NP problems. If we can solve K in polynomial time, then we can solve **ALL** NP problems in polynomial time, and thus NP=P.

Conversely, if indeed $NP \neq P$, then a NP-complete problem can not be solved in polynomial time.

Now the question is to find these NP-complete problems.

Theorem

If $K \in NP$, and $K \geq Y$ where Y is NP-complete.
Then K is NP-complete.

Another NP problem **SAT-3-CNF**

definition: A *literal* is a Boolean or its negation or 1 or 0.
A 3-*clause* is a disjunction ("or") of 3 literals.
A 3-*CNF* of is a conjunction ("and") of 3-clauses.

e.g. $A= (\bar{a}+\bar{b}+c)(a+\bar{c}+d)(a+0+0)$

$B= (\bar{a}+\bar{b}+c)(a+b+0)(c+0+0)$
The input is a 3-CNF with *n* variables. Is there a way to assign 0/1 (TRUE/FALSE) to the variables so that the formula is 1 (TRUE).

in the above eg. by assigning a=1, b=0, c=0, d=0, then A is 1. Equation B is always 0.

What can we do if a problem is NP-hard

1. Fast algorithm that find the solution for small input.
2. Algorithm that find approximate solution.
3. Algorithm that find solution for special type of instances.

Approximation Solution / Algorithm

Let OPT be the cost of the optimal solution.
If we can find a solution with cost APR, such that

$APR < \varepsilon\, OPT$, where ε is a constant greater than 1.

then we say that the solution is a ε-approximation solution, and the algorithm that find the approximation solution is called the ε-approximation algorithm.

Examples (approximation algo)

Traveling Salesman problem.

Input: A complete undirected graph G=<V,E> that has nonnegative cost c(u,v) associated to each edge (u,v).

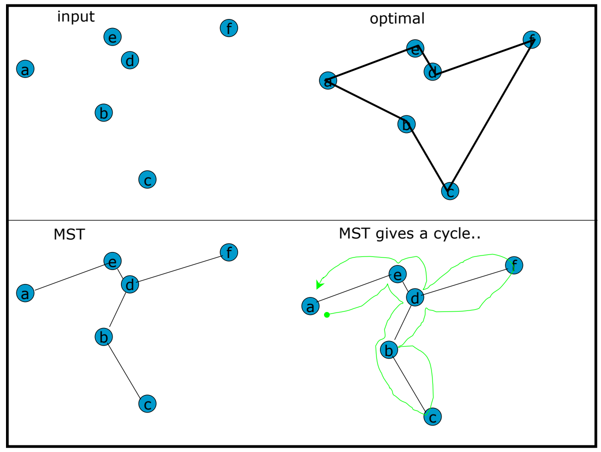
Output: A cycle (that is, a path that start and end at the some vertex) that visits each vertex once, with minimum cost.

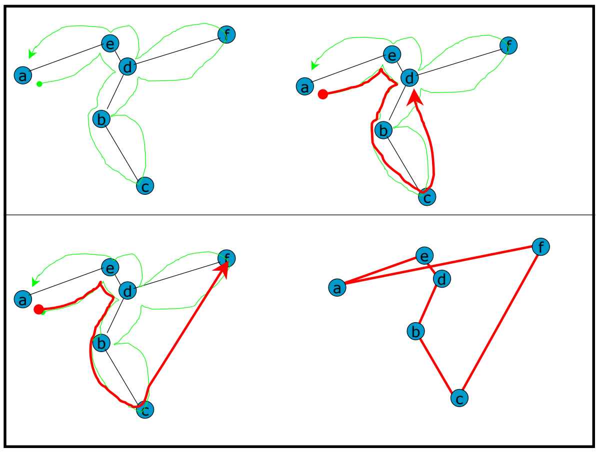
This is a NP-hard problem. Let us now look at a special case where the graph is an **Euclidean Graph**, and give a 2-approximation algorithm.

Traveling salesman on Euclidean Graph (triangluar inequality)

Algorithm Approx-TSP

1. Find the minimum spanning tree.
   (Note that the MST can be converted into a cycle.)
2. Randomly select a vertex and designate it as the root.
3. Do a preorder traversal of the tree.
4. Return the cycle that visits the vertices in the order computed in step 3.





Claim: Approx-TSP is a 2-approximation algorithm.

Let H* be the optimal cycle, and let T be the MST.
By removing any edge from H*, it become a spanning tree. Thus

cost (T) $\leq$ cost (H*).

The cycle obtained from T in step 1 traverses every edges in T twice.
Let W be this cycle. Clearly
cost (W) = 2 cost (T).

Note that W is not a solution, because vertices are visited twice.
Now, just remove the repeating vertices. If W visits the vertices in this order..

...... v_1, v_2, v_3, ...

By removing v_2, we will visit v_3 after visiting v1,. That is, the edge from (v_1,v_2) and (v_2,v_3) will be replace by the edge (v_1,v_3).
By triangular inequality,

the length of $(v_1,v_3) \leq$ lenght of (v_1,v_2) + lenght of (v_2,v_3) .

Let H be the cycle obtained by removing all repeating vertices.
We have cost (H) $\leq$ cost (W) = 2 cost (T) < 2 cost (H*).
Thus cost (H) $\leq$ 2 cost (H*)

Remark on Approximation algorithm

- Note, however, that there are problem that does not have approximation algorihtm (unless P=NP).

For example, we can prove that, unless P=NP, TSP on general graph does not have an c-approximation algorithm, where c is a constant.