Intro to Sampling Theory

Sampling Theory

- The world is continuous
- Like it or not, images are discrete.
  - We work using a discrete array of pixels
  - We use discrete values for color
  - We use discrete arrays and subdivisions for specifying textures and surfaces
- Process of going from continuous to discrete is called sampling.

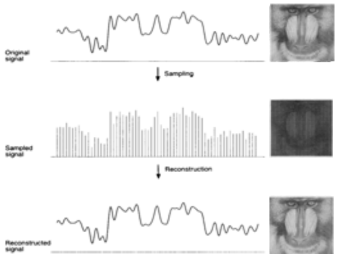
Sampling Theory

- Signal - function that conveys information
  - Audio signal (1D - function of time)
  - Image (2D - function of space)
- Continuous vs. Discrete
  - Continuous - defined for all values in range
  - Discrete - defined for a set of discrete points in range.

Sampling Theory

- Point Sampling
  - start with continuous signal
  - calculate values of signal at discrete, evenly spaced points (sampling)
  - convert back to continuous signal for display or output (reconstruction)

Sampling Theory



Foley/VanDam

Sampling Theory

- Sampling can be described as creating a set of values representing a function evaluated at evenly spaced samples

$$f_n = f(i\Delta) \qquad i = 0,1,2,\ldots,n$$

Δ = interval between samples = range / n.

Sampling Theory

- Sampling Rate = number of samples per unit

$$f = \frac{1}{\Delta}$$

- Example -- CD Audio
  - sampling rate of 44,100 samples/sec
  - Δ = 1 sample every $2.26\text{x}10^{-5}$ seconds

Issues:

- Important features of a scene may be missed
- If view changes slightly or objects move slightly, objects may move in and out of visibility.
- To fix, sample at a higher rate, but how high does it need to be?

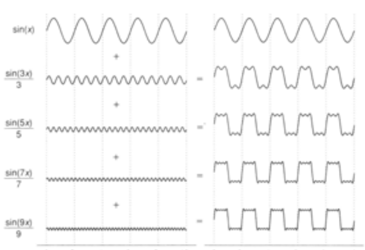
Sampling Theory

- Rich mathematical foundation for sampling theory
- Hope to give an "intuitive" notion of these mathematical concepts

Sampling Theory

- Spatial vs frequency domains
  - Most well behaved functions can be described as a sum of sin waves (possibly offset) at various frequencies
  - ***Frequency specturm*** - a function by the contribution (and offset) at each frequency is describing the function in the frequency domain
  - Higher frequencies equate to greater detail

Sampling Theory



Foley/VanDam

Sampling Theory

- Nyquist Theorum
  - A signal can be properly reconstructed if the signal is sampled at a frequency (rate) that is greater than *twice* the highest frequency component of the signal.
  - Said another way, if you have a signal with highest frequency component of f_h, you need at lease $2f_h$ samples to represent this signal accurately.

Sampling Theory

- Example -- CD Audio
  - sampling rate of 44,100 samples/sec
  - Δ = 1 sample every 2.26×10^{-5} seconds
- Using Nyquist Theorem
  - CDs can accurately reproduce sounds with frequencies as high as 22,050 Hz.

Sampling Theory

- Aliasing
  - Failure to follow the Nyquist Theorum results in *aliasing*.
  - Aliasing is when high frequency components of a signal appear as low frequency due to inadequate sampling.
- In CG:
  - Jaggies (edges)
  - Textures
  - Missed objects

Sampling Theory

- Aliasing - example

Foley/VanDam(628)

High frequencies masquerading as low frequencies

Sampling Theory

- Annoying Audio Applet
  - http://ptolemy.eecs.berkeley.edu/eecs20/week13/aliasing.html

Anti-Aliasing

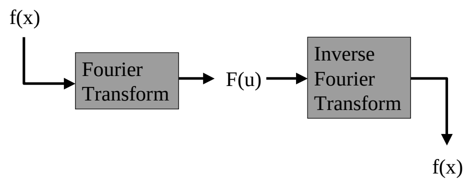
- What to do in an aliasing situation
  - Increase your sampling rate (supersampling)
  - Decrease the frequency range of your signal (Filtering)
- How do we determine the contribution of each frequency on our signal?

Fourier analysis

- Given f(x) we can generate a function F(u) which indicates how much contribution each frequency u has on the function f.
- F(u) is the Fourier Transform
- Fourier Transform has an inverse

Sampling Theory

- Fourier Transforms



Sampling Theory

- The Fourier transform is defined as:

$$F(u) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi ut} dt$$

Note: the Fourier Transform is defined in the complex plane

Sampling Theory

- The Inverse Fourier transform is defined as:

$$f(t) = \int_{-\infty}^{\infty} F(u) e^{i2\pi ut} du$$

Sampling Theory

- How do we calculate the Fourier Transform?
  - Use Mathematics
  - For discrete functions, use the Fast Fourier Transform algorithm (FFT)
- Can filter the transform to remove offending high frequencies - partial solution to anti-aliasing

Anti-aliasing -- Filtering

- Removes high component frequencies from a signal.
- Removing high frequencies results in removing detail from the signal.
- Can be done in the frequency or spatial domain

Getting rid of High Frequencies

- Filtering -- Frequency domain
  - Place function into frequency domain F(u)
  - Simple multiplication with *box filter* S(u), aka *pulse function*, band(width) limiting or low-pass filter.

$$S(u) = \begin{cases} 1, & \text{when } -k \le u \le k \\ 0, & \text{elsewhere} \end{cases}$$

  - Suppress all frequency components above some specified cut-off point *k*

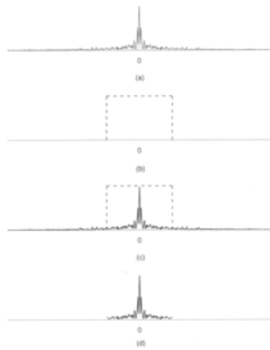
Filtering – Frequency Domain

Original Spectrum

Low-Pass Filter

Spectrum with Filter

Filtered Spectrum



Foley/VanDam(631)

Getting Rid of High Frequencies

- Filtering -- Spatial Domain
  - Convolution (* operator) - equivalent to multiplying two Fourier transforms

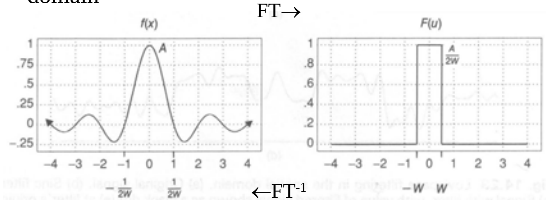
$$h(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(\tau) g(x-\tau) d\tau$$

Taking a weighted average of the neighborhood around each point of *f*, weighted by *g* (the ***convolution*** or ***filter kernel***) centered at that point.

Convultion *sinc* Function

- Convolving with a ***sinc*** function in the spatial domain is the same as using a box filter in the frequency domain

FT→



←FT^{-1}

Foley/VanDam (634)

Filtering using Convolution

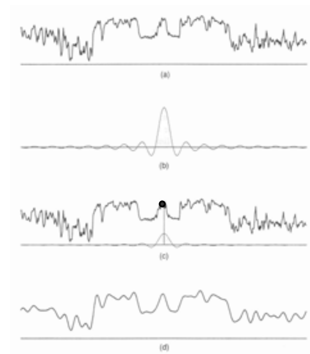
Original Spectrum

Sinc Filter

Spectrum with Filter

• value of filtered signal

Filtered Spectrum



Foley/VanDam (633)

Convolution

- Joy of Convolution applet

http://www.jhu.edu/~Esignals/convolve/index.html

Sampling Theory

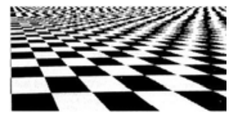
- Anti-aliasing -- Filtering
  - Removes high component frequencies from a signal.
  - Removing high frequencies results in removing detail from the signal.
  - Can be done in the frequency or spatial domain

Sampling Theory

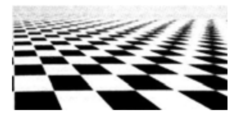
- 2D Sampling
  - Images are examples of sampling in 2-dimensions.
  - 2D Fourier Transforms provides strength of signals at frequencies in the horizontal and vertical directions

Sampling Theory

- 2D Aliasing



aliased image



anti-aliased image

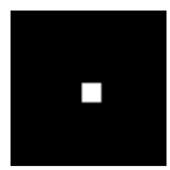
Foley/VanDam

Sampling Theory

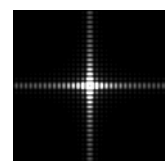
- 2D Fourier Transform

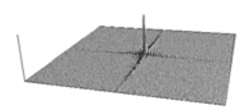
$$F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-i2\pi(ux+vy)} dxdy$$

Sampling Theory



→→→→

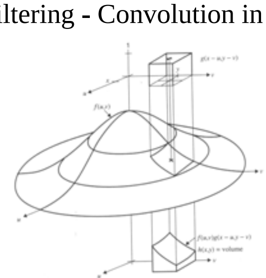




Castleman

Sampling Theory

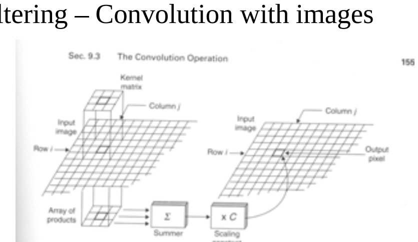
- Filtering - Convolution in 2D



Castleman

Sampling Theory

- Filtering – Convolution with images



Castleman

Sampling Theory

- Filtering – Convolution in frequency domain

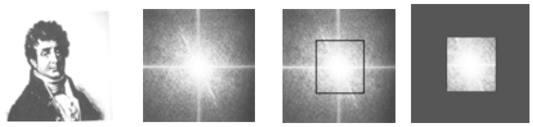


Image | 2D FFT | Filter out high frequencies | Filtered 2D FFT

Castleman

Anti-Aliasing – (Unweighted) Area Sampling

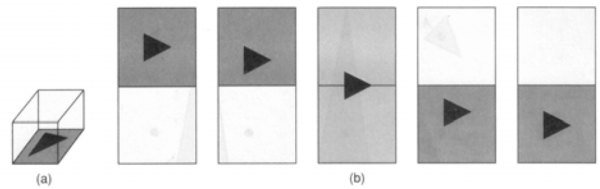


Fig. 14.11 Unweighted area sampling. (a) All points in the pixel are weighted equally. (b) Changes in computed intensities as an object moves between pixels.

Foley/VanDam (622)

Anti-Aliasing - Weighted Area Sampling

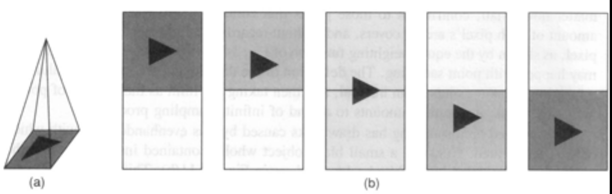


Fig. 14.12 Weighted area sampling. (a) Points in the pixel are weighted differently. (b) Changes in computed intensities as an object moves between pixels.

Foley/VanDam (622)

Anti-Aliasing - Weighted with Overlap

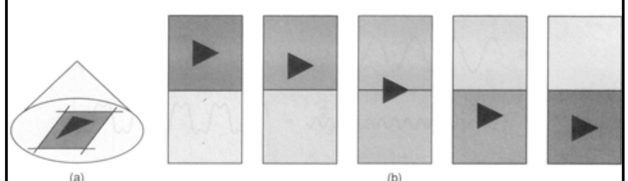
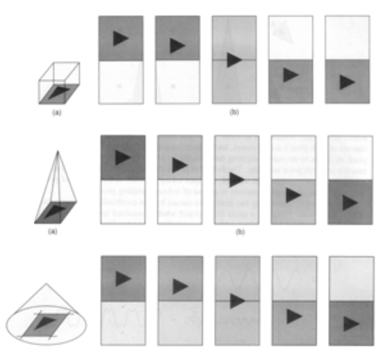


Fig. 14.13 Weighted area sampling with overlap. (a) Typical weighting function. (b) Changes in computed intensities as an object moves between pixels.

Foley/VanDam (622)



Area Sampling

Weighted Area Sampling

Weighted Area Sampling with Overlap

Foley/VanDam (622/3)

Other Anti-aliasing Methods

- ***Pre-filtering*** - filtering at object precision before calculating pixel's sample
- ***Post-filtering*** - supersampling (as we've seen)
- ***Adaptive supersampling*** - sampling rate is varied, applied only when needed (changes, edges, small items)
- ***Stochastic supersampling*** - places samples at stocastically determined positions rather than regular grid

Anti-Aliasing

- Applet

  http://www.nbb.cornell.edu/neurobio/land/OldStudentProjects/cs490-96to97/anson/AntiAliasingApplet/index.html

Sampling Theory

- Summary
  - Digital images are discrete with finite resolution…the world is not.
  - Spatial vs. Frequency domain
  - Nyquist Theorum
  - Convolution and Filtering
  - 2D Convolution & Filtering
  - Questions?

Sampling Theory

- Further Reading
  - Foley/VanDam – Chapter 14
  - *Digital Image Processing* by Kenneth Castleman
  - Glassner, Unit II (Book 1)

Remember

- Class Web Site:
  - http://www.cs.rit.edu/~jmg/cgII
- Any questions?