Computing and Counting the Longest Paths on Circular-Arc Graphs in Polynomial Time

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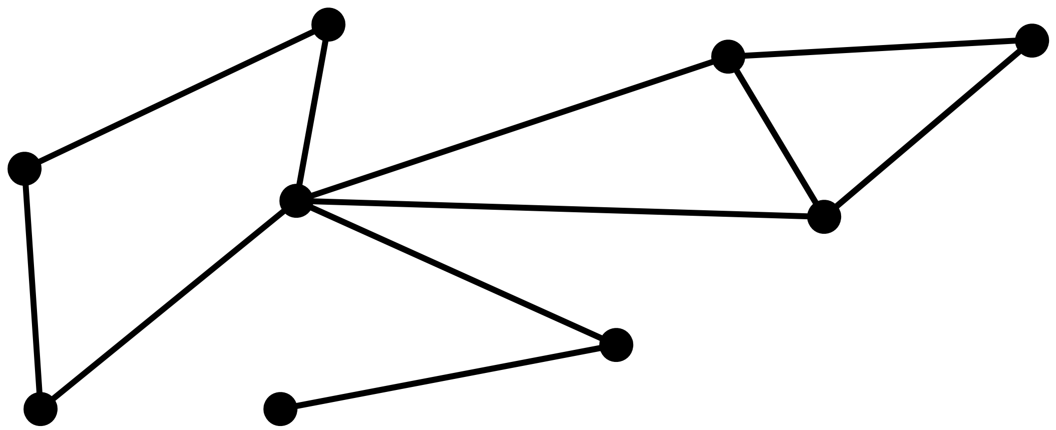
The Longest Path Problem

Input: an undirected graph G

Output: a path with the largest possible length

Example:



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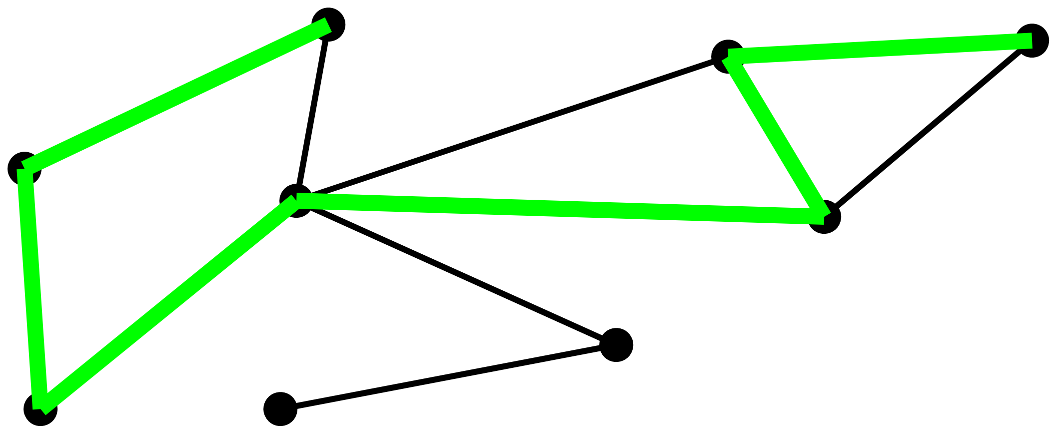

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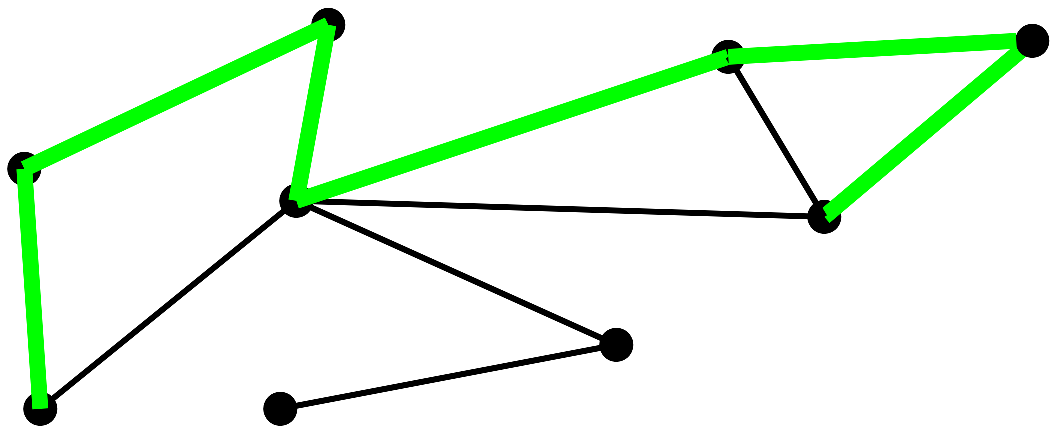

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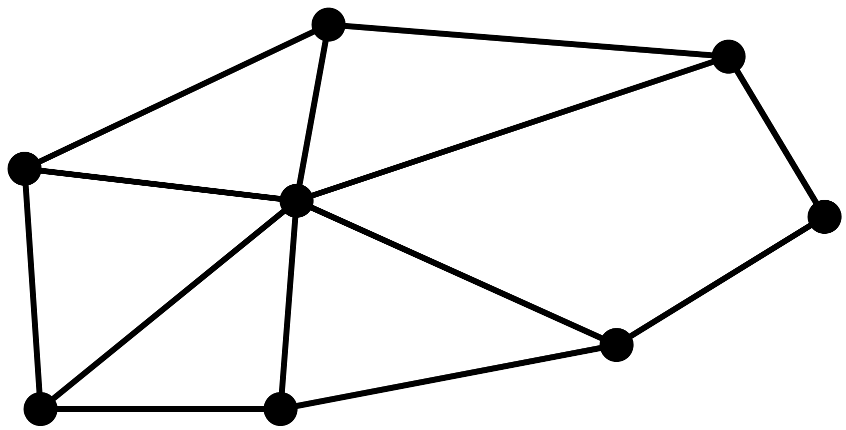

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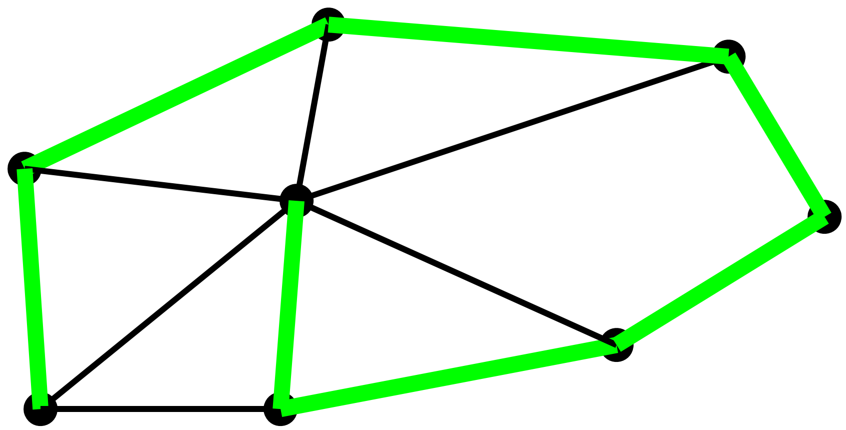


Hamiltonian path: a path going through all vertices

Note: Longest Path generalizes the Hamiltonian path problem, hence Longest Path is NP-hard

The Longest Path Problem

Hamiltonian and Longest Path are NP-hard on:

- general graphs
- planar graphs
- bipartite graphs
- split graphs
- ... [see, e.g., Mertzios PhD Thesis '09 for the references]

Inapproximability results [Karger-Motwani-Ramkumar '97]:

- For any $\epsilon \in (0,1)$, finding a path of length $n-n^{\epsilon}$ in a graph with a Hamiltonian path is NP-hard

- If P$\neq$NP, no constant-factor approx. for Longest Path

The Longest Path Problem - Related Works

Path problems in polynomial time:

- Ham. Path on
  - proper interval graphs [Bertossi '83]
  - interval graphs [Arikati-Rangan '90, Keil '85]
  - circular-arc graphs [Damaschke '93, Hung-Chang-Laio '09, Shih-Chern-Hsu '92]
  - cocomparability graphs [Hung-Chang '06, Hung-Chang-Laio '09]

The Longest Path Problem – Related Works

Path problems in polynomial time:

- Longest path – very recent:
    - trees [Bulterman-van der Sommen-Zwaan-Verhoeff-van Gasteren-Feijen '02]
    - weighted trees and block graphs [Uehara-Uno '04]
    - bipartite permutation graphs [Uehara-Valiente '07]
    - ptolemaic graphs [Takahara-Teramoto-Uehara '08]

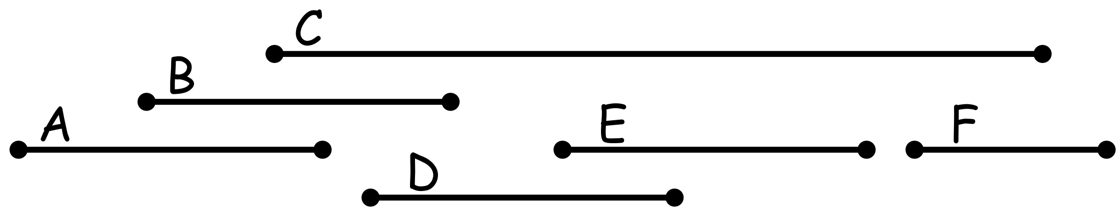
    - interval graphs [Ioannidou-Mertzios-Nikolopoulos '09]
    - cocomparability graphs [Ioannidou-Nikolopoulos '10 – $O(n^8)$, Mertzios-Corneil '10 – $O(n^4)$]

Examples of applications of Longest Path:
e.g., computational biology [Colinge-Bennett '07, Cuntz-Borst '07]

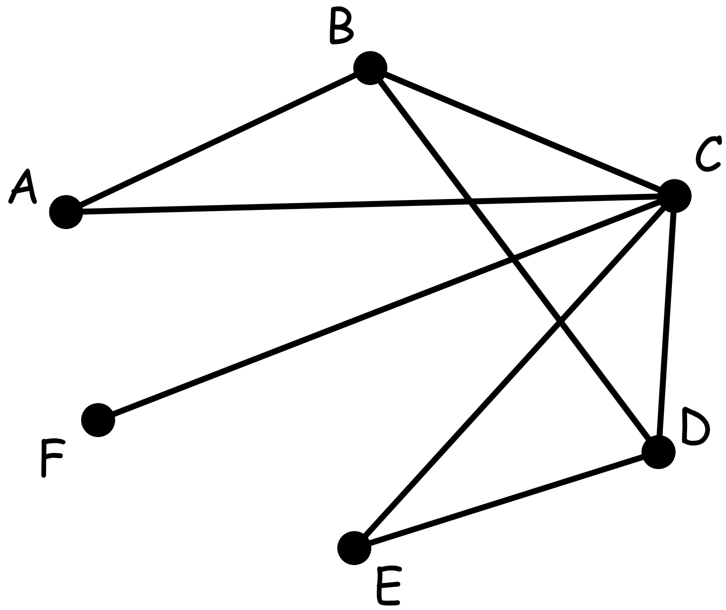
Interval vs. Circular-Arc Graphs

Interval graphs: intersection graphs of intervals

An interval graph representation:



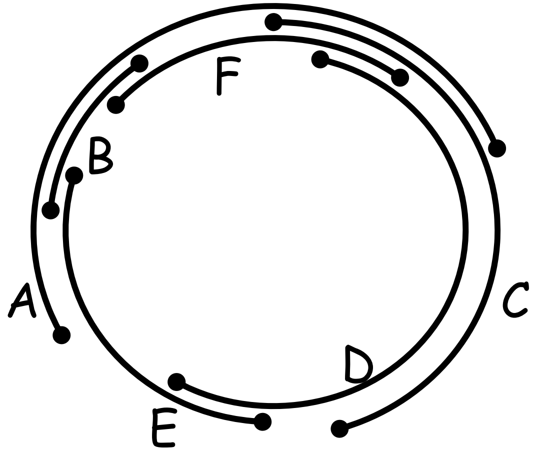
The corresponding graph:



Interval vs. Circular-Arc Graphs

Circular-arc graphs: intersection graphs of arcs on a circle

A circular-arc graph representation:



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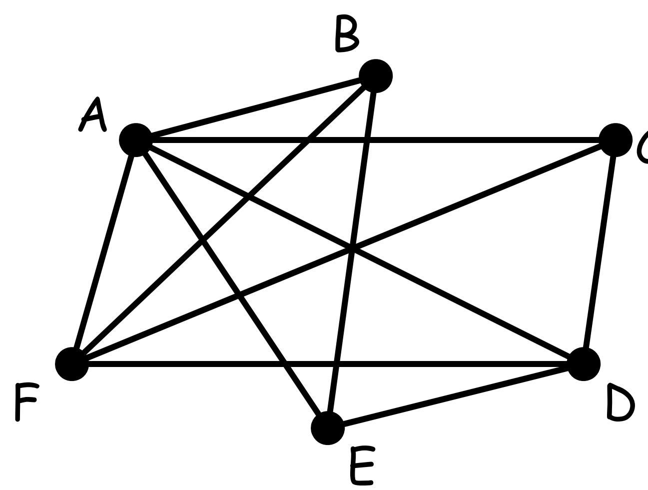


Interval vs. Circular-Arc Graphs

For some problems, interval graphs are very different from circular-arc graphs, e.g.:

Minimum chromatic number (coloring):

- polynomial-time for interval graphs [Garey-Johnson-Miller-Papadimitriou '80]
- NP-complete for circular-arc graphs [e.g., Golumbic '04]

Motivation for circular-arc graphs:

- scheduling periodic tasks

Our contributions

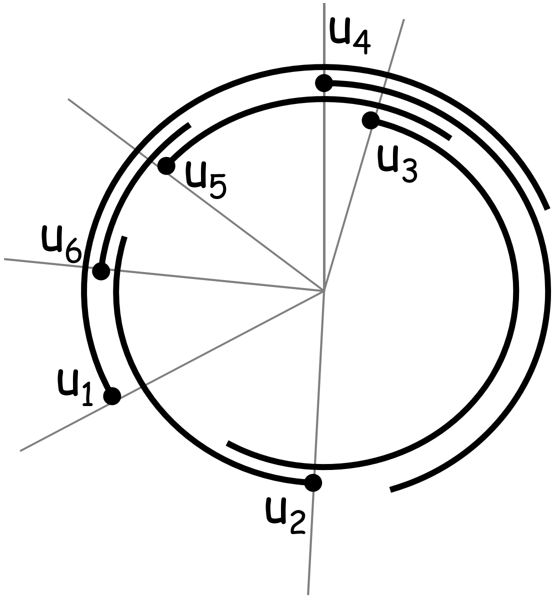
- Reduction from circular-arc graphs to interval graphs
- Simplification of the algorithm for interval graphs by Ioannidou, Mertzios, and Nikolopoulos '09
- Counting and sampling of ("normal") longest paths

Reduction: circular-arc → interval graphs

Some terminology:

- denoting arcs be their right end-points:

- right-end ordering of arcs:



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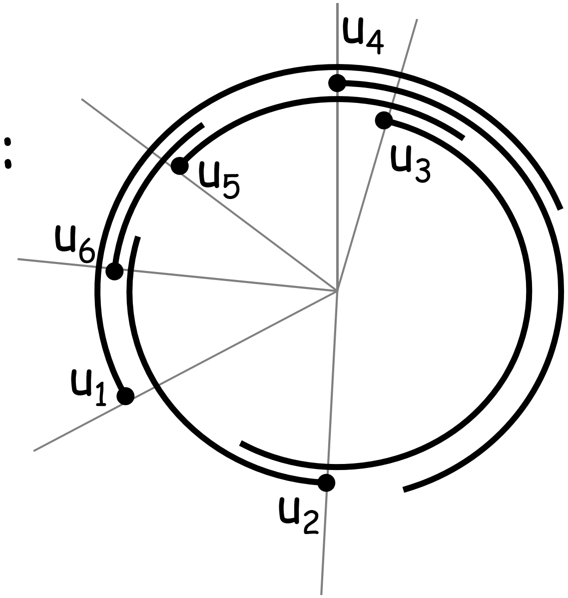
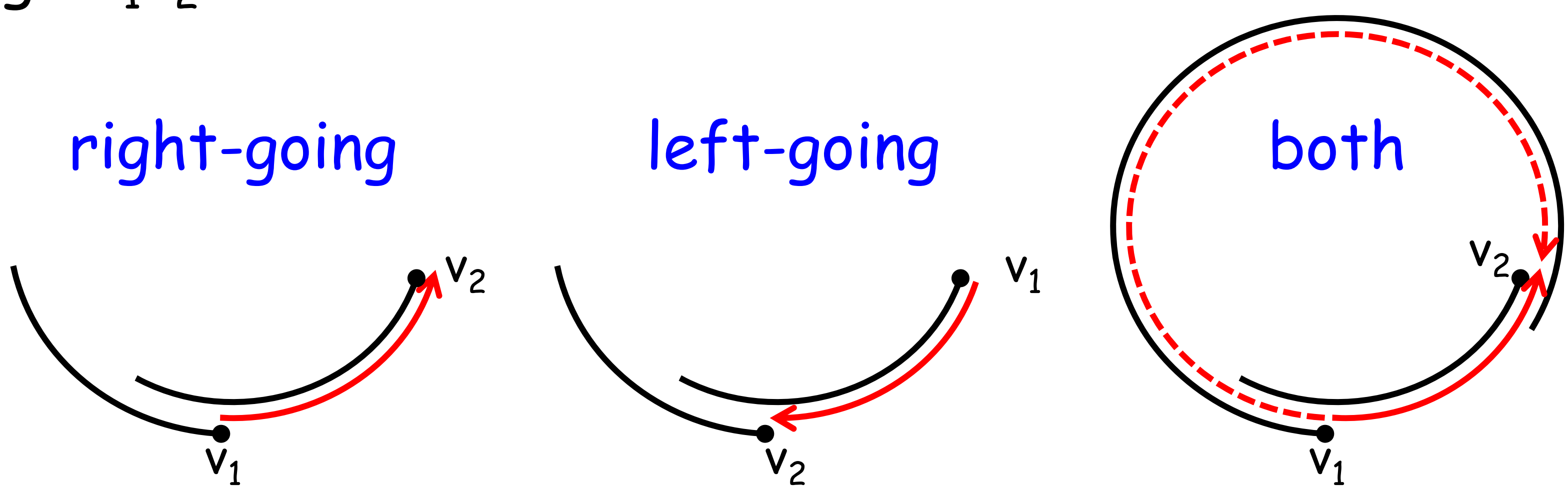


- edge v_1v_2 is:



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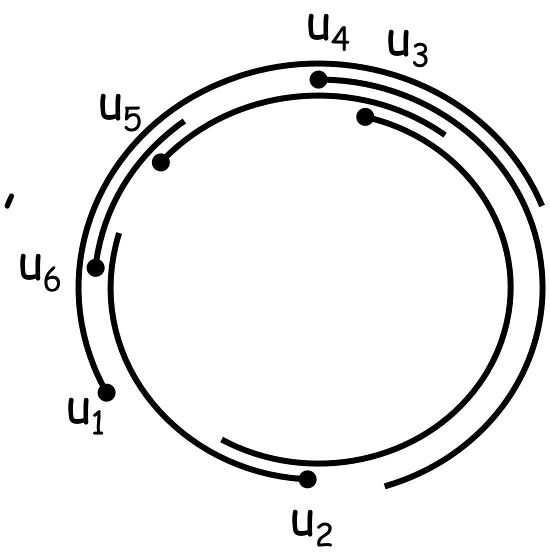
- path-arc representation:
  representing edges by the red arcs,
  e.g. path $u_6u_2u_3u_1u_4u_5$:



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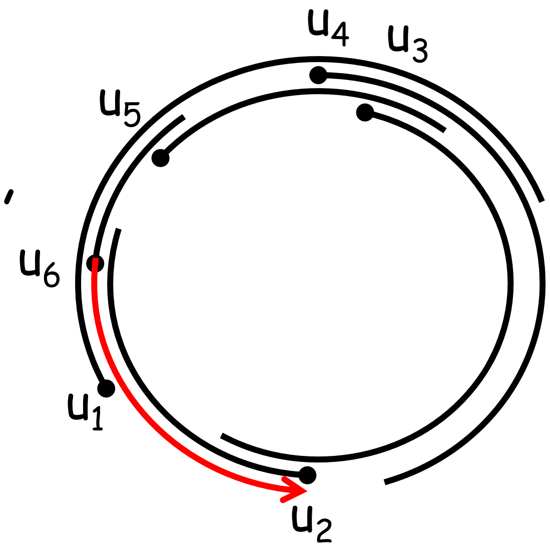

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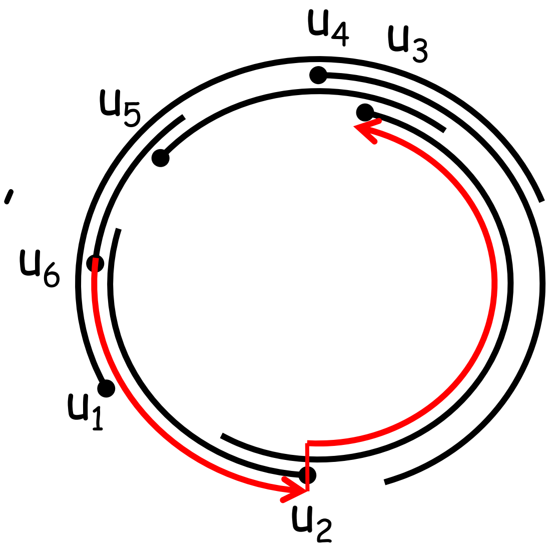

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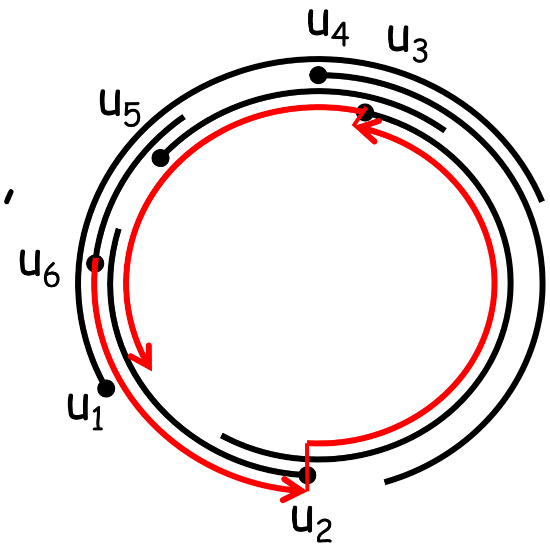

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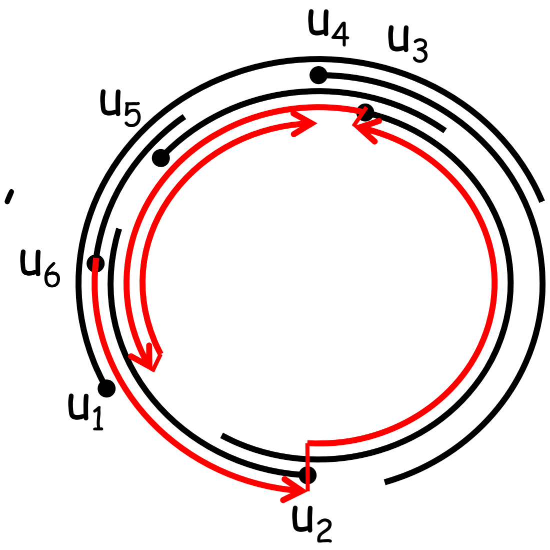

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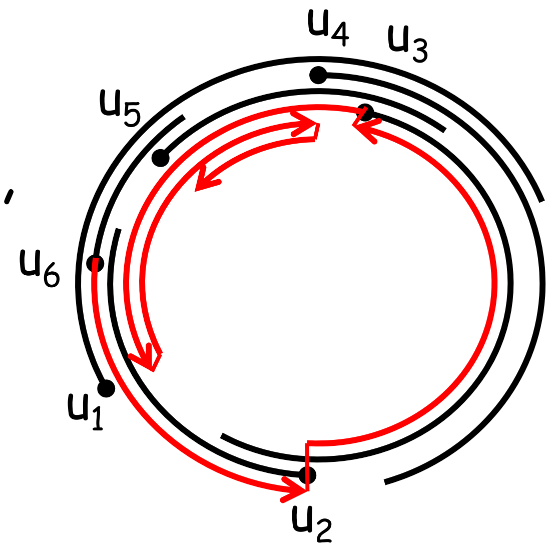

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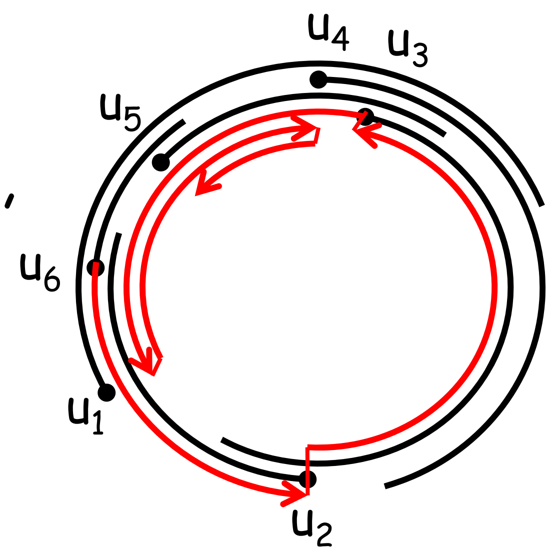

Idea: for **every** path P, there exists a path P' on the same set of vertices with path-arc representation that does not cover the entire circle

Reduction: circular-arc → interval graphs

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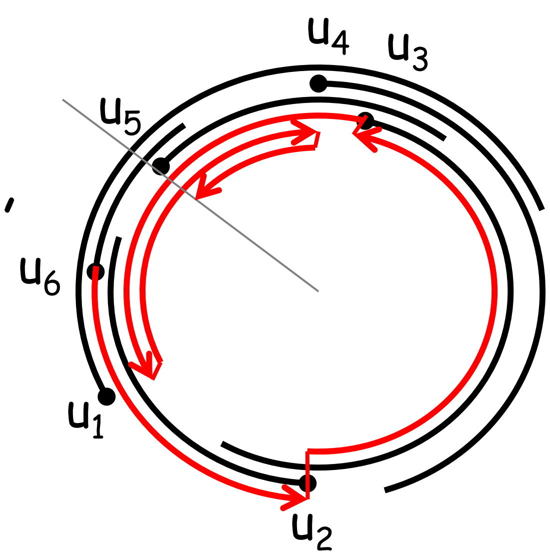
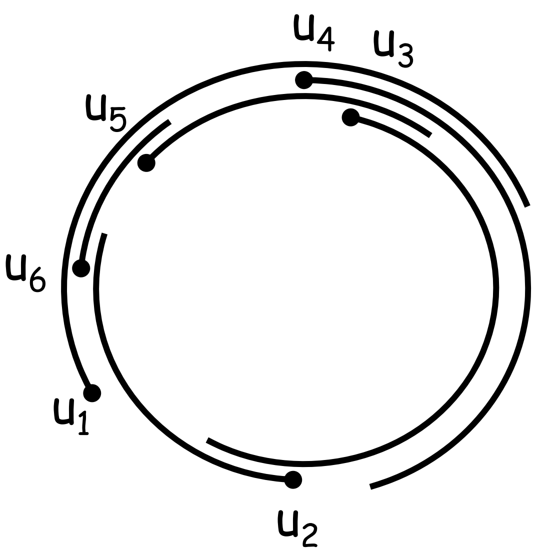


- given a path-arc representation:
  - right-cut and left-cut at a vertex u:
    number of right- and left-going red arcs containing u,
    e.g. for u_5: right-cut = left-cut = 1
  - cut at a vertex u = right-cut + left-cut

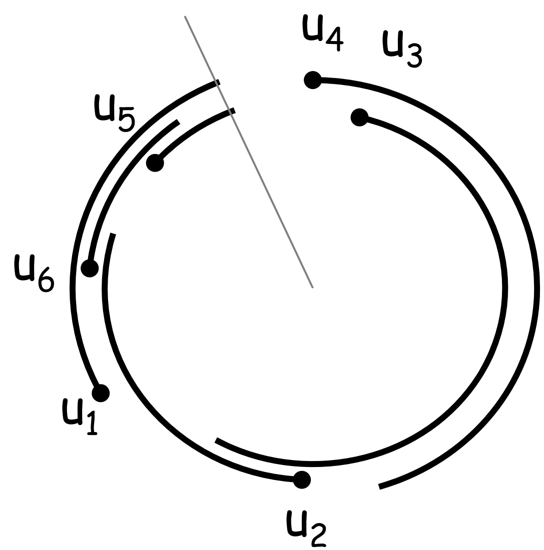
Reduction: circular-arc → interval graphs

Some terminology:

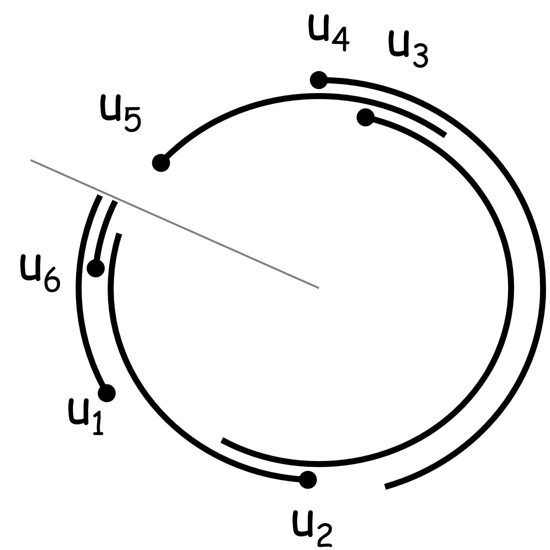
- cutting the circle before/after a vertex, e.g. for u_5 :



original representation



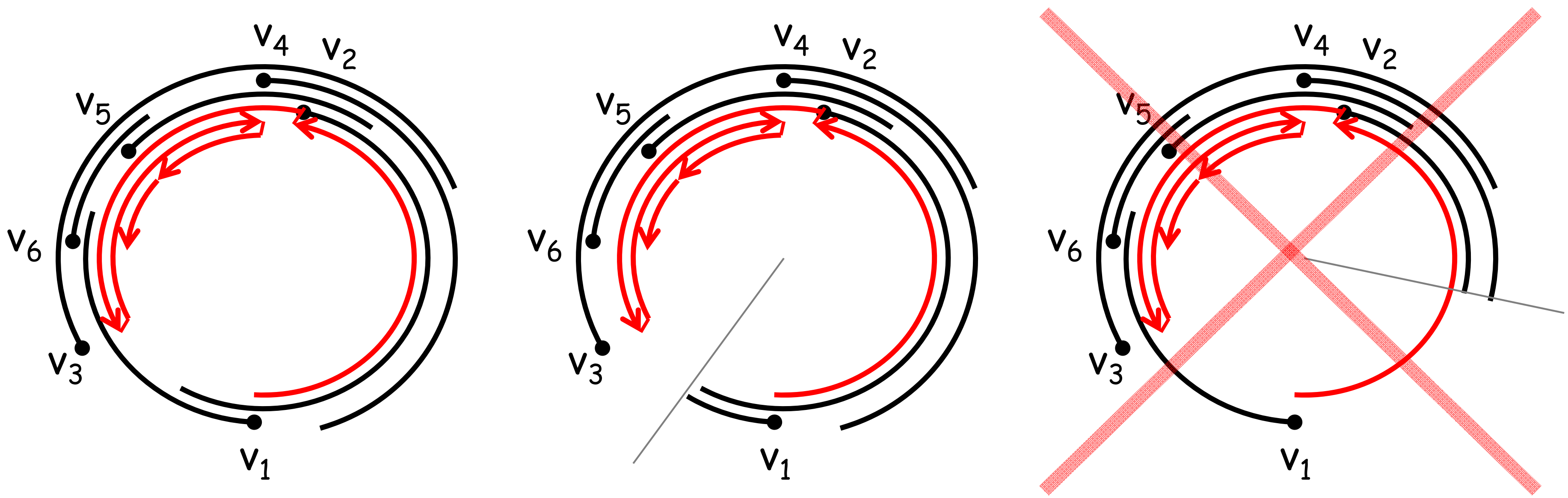
representation for cut before u_5 → get graph G_{u_5}



representation for cut after u_5

Reduction: circular-arc → interval graphs

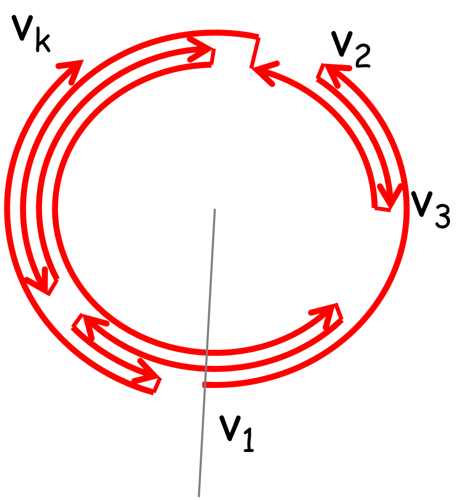
Lemma 1: Let $v_1v_2 \ldots v_k$ be a path with a path-arc representation such that cut(v_1) = 0. Then at least one of the graphs obtained from cutting before v_1 or after v_1 contains the path $v_1v_2 \ldots v_k$.



Reduction: circular-arc → interval graphs

Lemma 2: Let $v_1 v_2 \ldots v_k$ be a path such that $\sum_{i=1\ldots k}$ cut(v_i) is the smallest possible across all paths on $\{v_1, v_2, \ldots, v_k\}$ (and their representations). Then, right-cut(v_1) = 0.

Proof: By contradiction, right-cut(v_1)>0.



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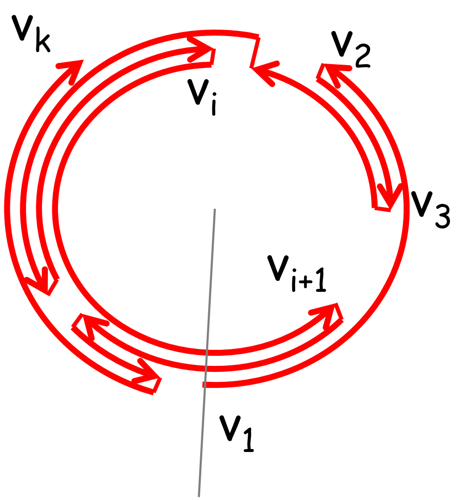

Let v_iv_{i+1} be a right-going edge with red arc containing v_1.

Reduction: circular-arc → interval graphs

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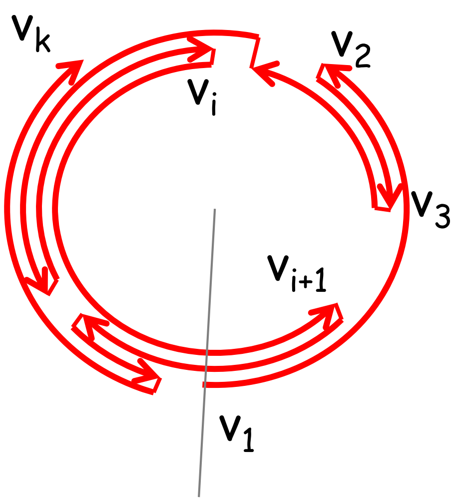

Let $v_i v_{i+1}$ be a right-going edge with red arc containing v_1.

Consider path $v_i v_{i-1} v_{i-2} \ldots v_1 v_{i+1} v_{i+2} \ldots v_k$.

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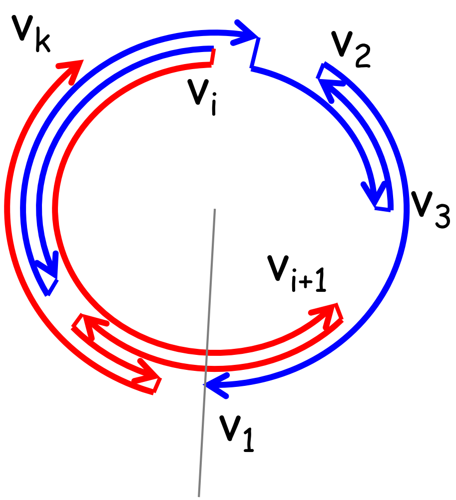

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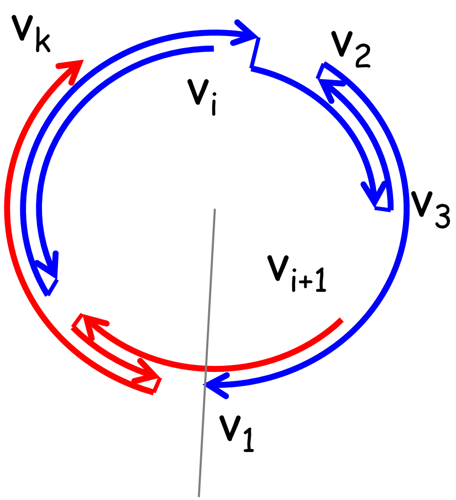

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Edge $v_i v_{i+1}$ disappeared.

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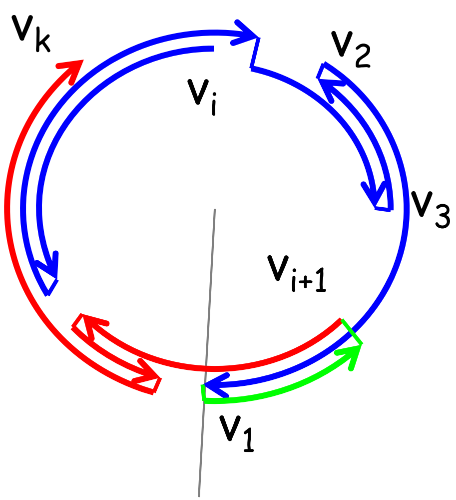

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Consider path $v_iv_{i-1}v_{i-2} \ldots v_1v_{i+1}v_{i+2} \ldots v_k$.

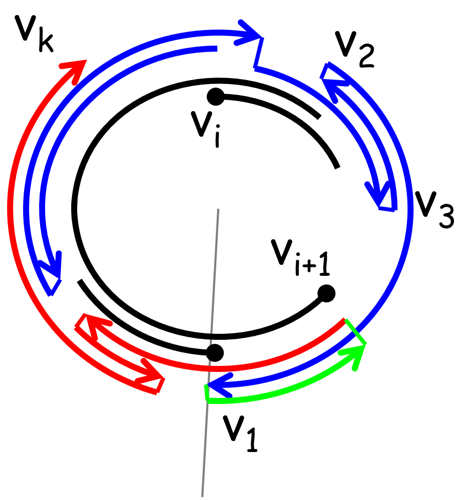
Edge v_iv_{i+1} disappeared.

What about edge v_1v_{i+1} ?

Reduction: circular-arc → interval graphs

Lemma 2: Let $v_1v_2\ldots v_k$ be a path such that $\sum_{i=1\ldots k}$ cut(v_i) is the smallest possible across all paths on $\{v_1,v_2,\ldots,v_k\}$ (and their representations). Then, right-cut(v_1) = 0.

Proof: By contradiction, right-cut(v_1)>0.



Claim: edge v_1v_{i+1} exists.

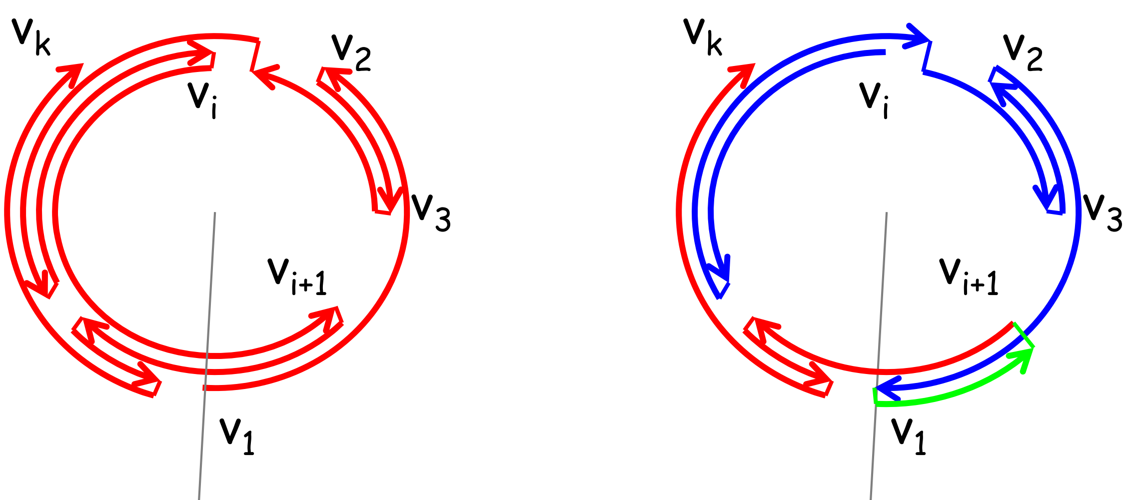
[Follows from existence of right-going edge v_iv_{i+1}.]

Reduction: circular-arc → interval graphs

Lemma 2: Let $v_1 v_2 \ldots v_k$ be a path such that $\sum_{i=1\ldots k} \text{cut}(v_i)$ is the smallest possible across all paths on $\{v_1, v_2, \ldots, v_k\}$ (and their representations). Then, right-cut(v_1) = 0.

Proof: By contradiction, right-cut(v_1)>0.

The new path decreases $\sum_{i=1\ldots k} \text{cut}(v_i)$, a contradiction. □



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Corollary 2: Under the same conditions, left-cut(v_k)=0.

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Corollary 2: Under the same conditions, left-cut(v_k)=0.

Lemma 3: Under the same conditions, cut(v_1)=0 or cut(v_k)=0.

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Corollary 2: Under the same conditions, left-cut(v_k)=0.

Lemma 3: Under the same conditions, cut(v_1)=0 or cut(v_k)=0.

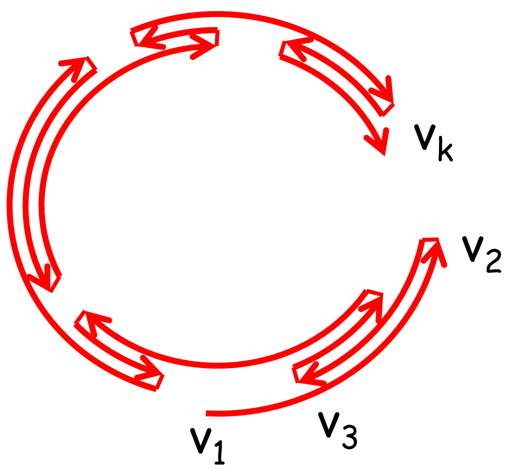
Theorem 1: For any path P there exists a path P′ on the same vertex set and a vertex v such that the path P′ is a path in the interval graph obtained by cutting the circle before v.

Reduction: circular-arc → interval graphs

Lemma 3: Let $v_1v_2\ldots v_k$ be a path such that $\sum_{i=1\ldots k}$ cut(v_i) is the smallest possible across all paths on $\{v_1,v_2,\ldots,v_k\}$ (and their representations). Then, cut(v_1)=0 or cut(v_k)=0.

Proof idea:

If v_1's arc goes right and v_k's arc comes from right, we get:



The full proof contains significant case analysis.

The idea of the simplified algorithm

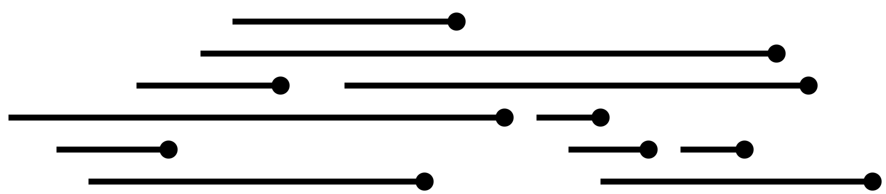
Some terminology:

- normal paths in interval graphs:

  $v_1 v_2 \ldots v_k$ is normal if

  - v_1 is the left-most vertex

  - for every i, v_i is the left-most neighbor of v_{i-1} out of all neighbors of v_{i-1} among $\{v_i, v_{i+1}, v_{i+2}, \ldots, v_k\}$



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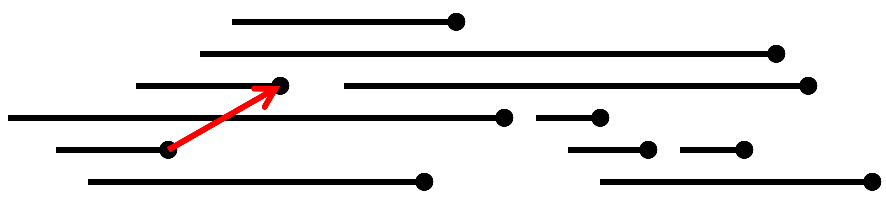

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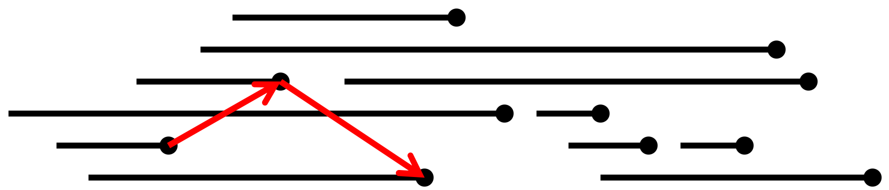

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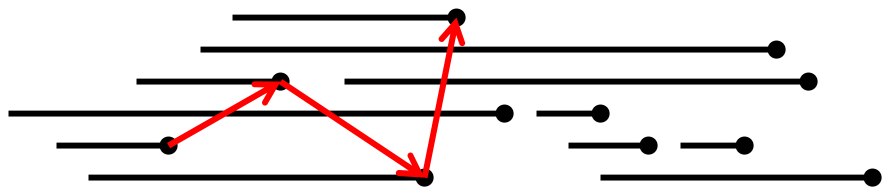

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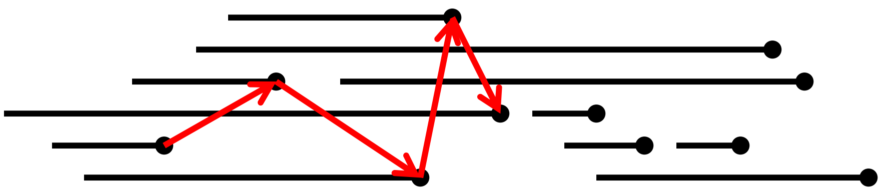

The idea of the simplified algorithm

Some terminology:

- normal paths in interval graphs:

  $v_1 v_2 \ldots v_k$ is normal if

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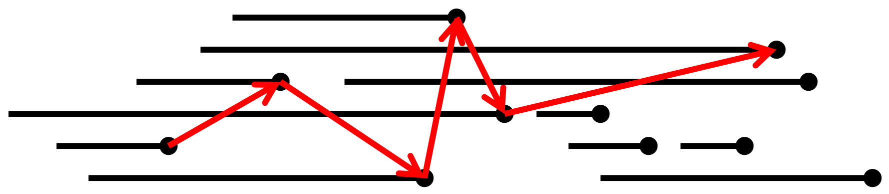

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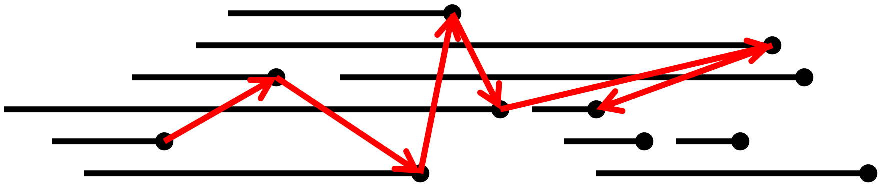

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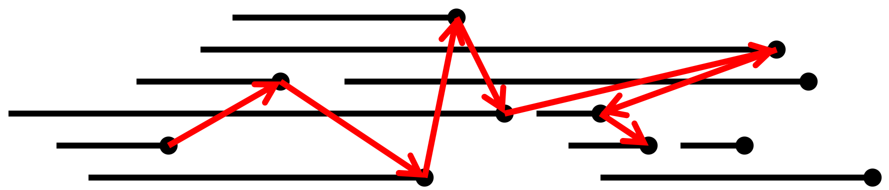

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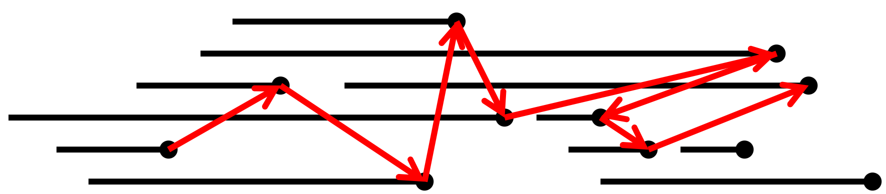

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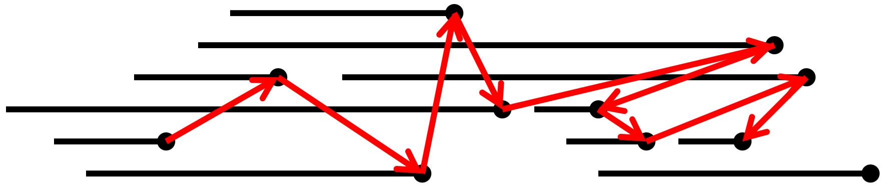

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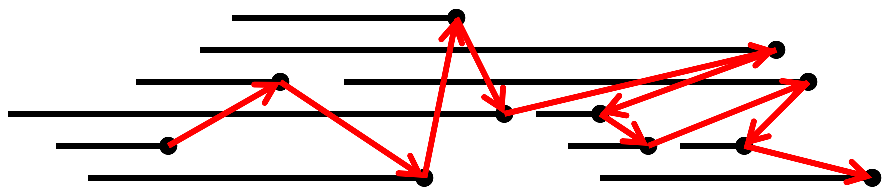

The idea of the simplified algorithm

Some terminology:

- normal paths in interval graphs [Ioannidou-Mertzios-Nikolopoulos '09]:

  $v_1 v_2 \ldots v_k$ is normal if

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  - for every i, v_i is the left-most neighbor of v_{i-1} out of all neighbors of v_{i-1} among $\{v_i, v_{i+1}, v_{i+2}, \ldots, v_k\}$

Note:
similar notions introduced in [Damaschke '93, Keil '85]

The idea of the simplified algorithm

Some terminology:

- normal paths in circular-arc graphs:

  $v_1v_2 \ldots v_k$ is normal if it is normal in the interval graph obtained by cutting the circle before a vertex u

Theorem 2: For every path P in a circular-arc graph there exists a normal path P' on the same vertex set as P.

The idea of the simplified algorithm

Algo for interval graphs [Ioannidou-Mertzios-Nikolopoulos '09]:

- runs in time $O(n^4)$

- for technical reasons uses special dummy nodes that do not (much) modify the length of the longest path

Algo for circular-arc graphs:

- runs in time $O(n^4)$ [avoids the extra cost of the reduction]

- does not need the dummy nodes; as a byproduct of this simplification, it can be used for counting the normal paths [exactly for interval graphs, n-approximation for circular-arc graphs]

The idea of the simplified algorithm

Algo for circular-arc graphs - idea:

- dynamic programming
- need the following:
  - $G_i(j)$:= induced subgraph of G_{u_i} with vertices $\{u_i, u_{i+1}, \ldots, u_j\}$ where $i,j \in \{1,2,\ldots,n\}$
  - $G(i,j)$:= induced subgraph of G with vertices $\{u_i, u_{i+1}, \ldots, u_j\} \setminus \{u_k \mid u_k$ contains the right endpoint of $u_{i-1}\}$ where $i,j \in \{1,2,\ldots,n\}$ and $j \neq i-1 \mod n$
  - $\ell_i(u_k, j)$:= the length of a longest normal path of $G_i(j)$ with u_k as its last vertex
  - $\ell(u_k, i, j)$:= the length of a longest normal path of $G(i,j)$ with u_k as its last vertex

The idea of the simplified algorithm

Algo for circular-arc graphs - idea:

- why $O(n^4)$:
  - two loops to go through all i,j pairs
  - one loop for k
  - one loop to consider an intermediate "joining" vertex that connects two shorter paths

- can keep track of the number of all normal paths corresponding to $\ell_i(u_k,j)$ and $\ell(u_k,i,j)$ $\rightarrow$ a counting algorithm

Counting and sampling

Counting (and sampling) of normal paths:

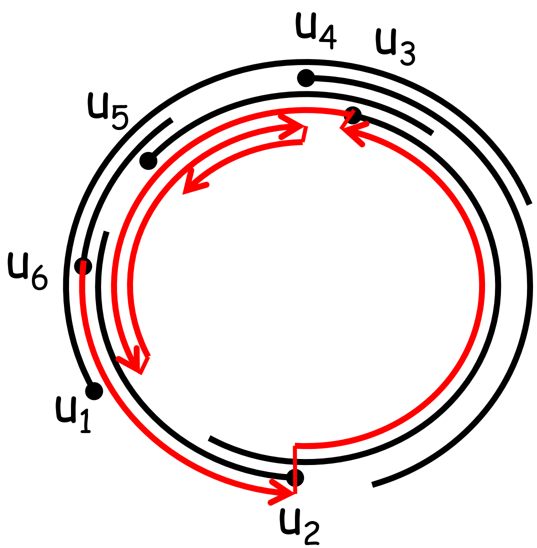
- in $O(n^4)$:
  - exact for interval graphs
  - n-approximation for circular-arc graphs
- for some graphs the number of normal paths can be exponentially large

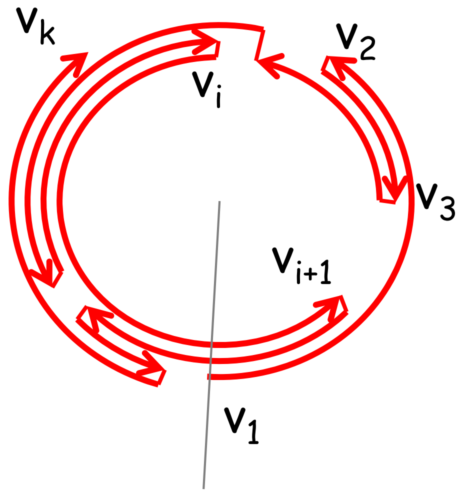
Counting/sampling of paths considered in other works:

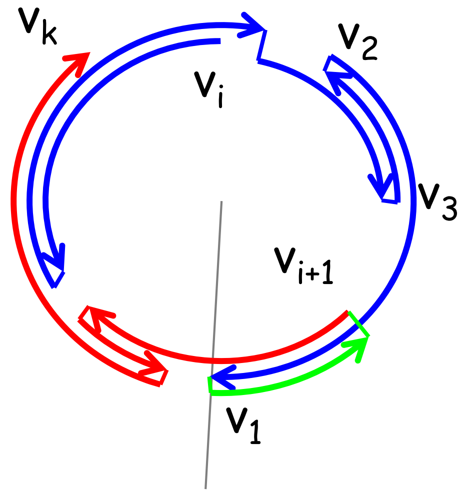
- #P-complete for general graphs [Dyer-Frieze-Jerrum '94]
- approx for special graph classes:
dense graphs [Dyer-Frieze-Jerrum '94], nearly regular [Frieze '00]
- self-avoiding walks in lattice graphs [Randall-Sinclair '00]

Open problems

Highlights from this talk:







Open problems:

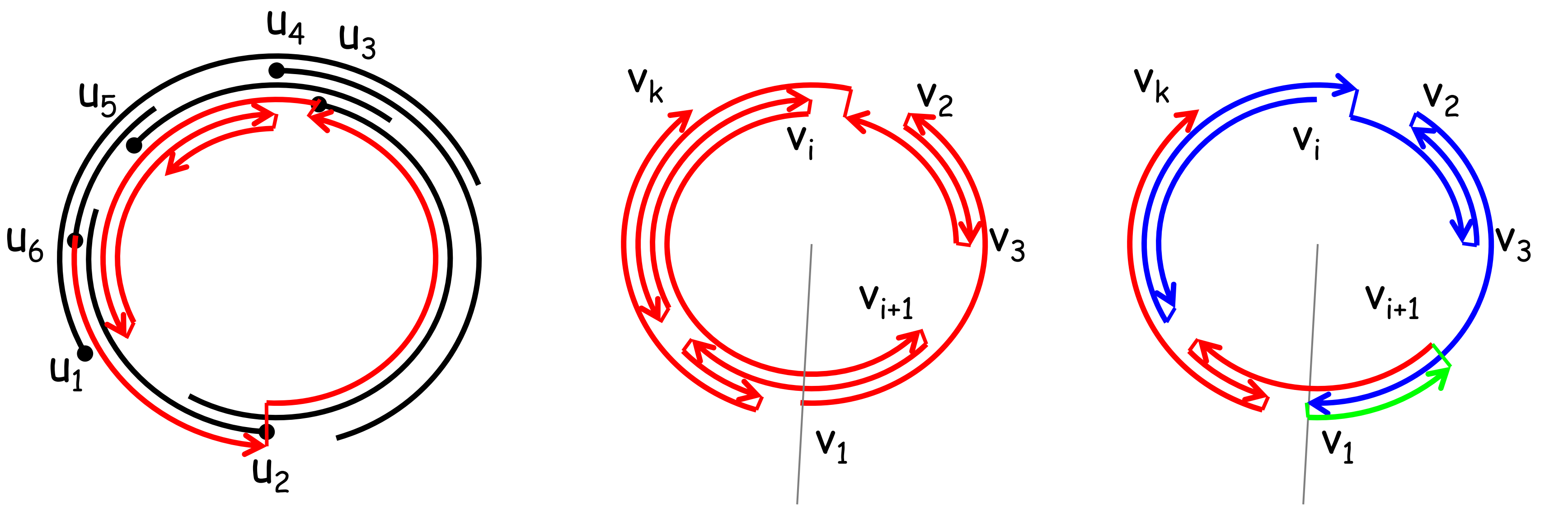
- count all longest paths in interval/circular-arc graphs

- improve the approximation ratio for counting normal paths in circular-arc graphs

Open problems

Highlights from this talk:

Thanks for your attention ☺



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- improve the approximation ratio for counting normal paths in circular-arc graphs