Graph Reconstruction
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• Called the major open problem in the field of Graph Theory
• Taking a tour through the topic
  – Introduction to Graph Reconstruction
  – Reconstructible classes
  – $k$ reconstruction
  – Complexity results
  – Reconstruction numbers
  – Open questions
Background

- Fundamental datatype is finite undirected simple graphs
- $V(G) = \text{Vertices of } G$  $E(G) = \text{Edges of } G$
- $\text{deg}(v) = \text{degree of vertex } v$
  - number of edge incident to vertex $v$
- $G-v = \text{graph } G \text{ with vertex } v \text{ removed}$
- $G-e = \text{graph } G \text{ with edge } e \text{ removed}$
- $\overline{G} = \text{Complement of } G$
  - Same vertex set, edges complemented
- $K_n = \text{clique of size } n$
  - $n$ completely connected vertices
Graph Isomorphism

- Two graphs are isomorphic if they differ only in the labels of equivalent vertices
  - For our purposes isomorphic graphs are completely equivalent
  - Example of a set of isomorphic graphs:

- Graph Isomorphism decision problem is believed to be in NP-Intermediate
  - In NP, but not NP-complete, or in P
Graph Reconstruction Conjecture

• First proposed by Kelly and Ulam in 1941
• States that any graph on 3 or more vertices can be uniquely identified (up to isomorphism) by the multiset of its unlabeled 1-vertex-deleted subgraphs
• Believed to be true, but so far unproven in the general case
• Edge Reconstruction Conjecture proposed in 1964
  – graphs with 4 or more edges can be reconstructed from all 1-edge-deleted subgraphs
Basic Definitions

- **Card of G**: Graph G with one vertex removed
  - \( \begin{array}{ccc}
  0 & 1 \\
  2 & 3
  \end{array} \rightarrow \begin{array}{c}
  \end{array} \)

- **Deck(G) or D(G)**: multiset of all cards of G
  - \( \begin{array}{ccc}
  0 & 1 \\
  2 & 3
  \end{array} \rightarrow \begin{array}{c}
  \end{array} \)

- **Extensions(G)**: set of all graphs on \(|V(G)|+1\) vertices with G as a subgraph
  - \( \begin{array}{ccc}
  0 & 1 \\
  2 & 3
  \end{array} \rightarrow \begin{array}{c}
  \end{array} \)

- **Edge deleted versions**: \( \varepsilon \text{Card}, \varepsilon \text{Deck}, \varepsilon \text{Extensions} \)
Kelly’s Lemma

- $s(F,G)$ is the number of subgraphs of $G$ isomorphic to $F$
- $s(F,G)$ can be determined from $\text{Deck}(G)$ for any two graphs $F$ and $G$ where $|V(F)| < |V(G)|$

Proof:
- Each instance of $F$ in $G$ will appear $|V(G)| - |V(F)|$ times in $\text{Deck}(G)$, so:

$$s(F, G) = \frac{1}{|V(G)| - |V(F)|} \sum_{C \in D(G)} s(F, C)$$

- $s(F,G)$ can be similarly computed from $\mathcal{E}\text{Deck}(G)$
Corollaries to Kelly’s Lemma

• The $|E(G)|$ can be computed from Deck(G)
  
  $\sum_{C \in D(G)} \frac{1}{|V(G)| - |V(K_2)|} s(K_2, C) = \frac{1}{|D(G)| - 2} \sum_{C \in D(G)} |E(C)|$

• The degree sequence of G can be computed from Deck(G)
  
  $\deg(v) = s(K_2, G) - s(K_2, G - v)$
  
  $= \left( \frac{1}{|D(G)| - 2} \sum_{C \in D(G)} |E(C)| \right) - |E(C_v)|$
Regular Graphs Are Reconstructible

- Regular graphs are recognizable
  - Reconstructing the degree sequence tells us $G$ was regular of degree $r$

- Regular graphs are weakly reconstructible
  - Take any Card $C$ in Deck($G$), and add one vertex $v$, to make graph $C^*$
  - Connect $v$ to all vertices of degree $r-1$ in $C^*$
  - The new graph is isomorphic to $G$, and the only regular graph that could have $C$ in its deck
Other Reconstructible Classes

•Disconnected graphs
  – Easy to recognize: all but at most 1 card are disconnected

•Trees
  – First class to be shown as reconstructible
  – Doesn’t need all cards

•Maximal planar graphs
  – More recent result
Graph $k$-Reconstruction

- Introduced by Kelly in 1957 as a generalization to Graph Reconstruction
- Remove $k$ vertices or edges, instead of 1
- It is possible to construct a graph on $2k$ vertices which is not $k$-reconstructible for any $k \geq 1$
- There are few known results for $k > 1$
- Notational note: subscript signifies the number of vertex or edge deletions
Decision Problems

- **Legitimate Vertex Deck (LVD)**
  - Could the cards in the given deck have come from the same graph?
  - \( LVD_k = \{ <D> \mid \exists G, D = \text{Deck}_k(G) \} \)

- **Vertex Deck Check (VDC)**
  - Could the given graph produce the given deck?
  - \( VDC_k = \{ <D,G> \mid D = \text{Deck}_k(G) \} \)

- LED and EDC are the edge-deletion versions
Complexity

• In 2004 E. Hemaspaandra, L. Hemaspaandra, S. Radziszowski and R. Tripathi showed the following results:

\[ \forall k \geq 1 \quad GI \equiv_{iso}^{l} VDC_{k} \quad GI \equiv_{iso}^{l} EDC_{k} \]
\[ \forall k \geq 1 \quad GI \leq_{m}^{l} LVD_{k} \quad GI \equiv_{iso}^{l} LED_{k} \]
\[ \forall s \geq 2, \forall k \geq 1 \quad GI \equiv_{iso}^{p} s-VDC_{k} \quad GI \equiv_{iso}^{p} s-EDC_{k} \]
\[ \forall s \geq 2, \forall k \geq 1 \quad GI \leq_{m}^{l} s-LVD_{k} \quad GI \equiv_{iso}^{p} s-LED_{k} \]
\[ \forall k \geq 1 \quad GI \equiv_{iso}^{p} 2-LVD_{k} \]

• Notational note: s- prefix means that only s cards are given instead of the entire deck
Reconstruction Numbers

- In 1977 Harary and Planholt introduced the concept of reconstruction numbers
- $\exists \text{vrn}_k(G)$: the minimum number of $k$-vertex-deleted cards needed to reconstruct $G$ given your choice of cards
  - Also called the Ally Reconstruction Number
- $\forall \text{vrn}_k(G)$: the minimum number such that any $\text{Subdeck}_k$ of that size will reconstruct $G$
  - Also called the Adversarial Reconstruction Number
- $\exists \text{ern}_k$, $\forall \text{ern}_k$ are the $k$-edge-deleted versions
Reconstruction Numbers

- $\exists \text{vrn}_1$ is almost always 3
- $\exists \text{vrn}_k(G) = \exists \text{vrn}_k(\overline{G})$ and $\forall \text{vrn}_k(G) = \forall \text{vrn}_k(\overline{G})$
- $pK_n$ graphs have $\exists \text{vrn}_1 = n+2$
- In 2005 Brian McMullen produced the following results for his Master’s Project:

<table>
<thead>
<tr>
<th>Order</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
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<tbody>
<tr>
<td>$\exists \text{vrn}_1$</td>
<td>3</td>
<td>4</td>
<td>8</td>
<td>34</td>
<td>150</td>
<td>1,044</td>
<td>12,334</td>
<td>274,666</td>
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<td>5</td>
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<td>2</td>
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<td></td>
<td></td>
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<tr>
<td></td>
<td>6</td>
<td>2</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>266</td>
<td>45,186</td>
<td>6,054,148</td>
</tr>
<tr>
<td>$\forall \text{vrn}_1$</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>7</td>
<td>8</td>
<td>16</td>
<td>266</td>
<td>45,186</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>9</td>
<td>19</td>
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<td>12</td>
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<td>10,686</td>
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<td>4</td>
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</table>
Brian’s Algorithm

- Calculate:
  - \([d] = \text{Deck}(G)\)
  - \({ed} = \text{Extensions}(\ [d]\ )\)
  - \{[ded]} = \text{Deck}(\ {ed}\ )

- Use information to construct relation matrix describing how many times each card appears in each \(H\) in \{ed\}

- Search each subdeck for one not shared by any \(H\) in \{ed\}

```
1  2  3  4
1  0  1  0  1
2  0  1  1  1
3  0  1  1  0
4  1  1  1  0
5  0  1  1  0
6  1  0  0  1
7  1  0  0  0
8  0  0  0  1
9  0  0  0  1
```
Complexity of Brian’s Algorithm

- Canonizing generated graphs takes majority of time
  - n cards of order n-1 are canonized
    - of which $n/c_1$ are unique
  - $(n/c_1) \times 2^{(n-1)}$ extension graphs are canonized
    - of which $(n/c_1c_2) \times 2^{(n-1)}$ are unique
  - $(n/c_1c_2) \times 2^{(n-1)} \times n$ cards of extension graphs are canonized
- Overall: $O(n^2 \times 2^n)$ canonizations
- Generalizing to $k$-vertex reconstruction yields:
  - $O\left(n^{2k} 2(n-k) 2^{(k^2-k)/2}\right)$ worst case
## Runtime of Brian’s Algorithm

<table>
<thead>
<tr>
<th>Order</th>
<th>Number of unique graphs</th>
<th>1-vertex-reconstruction compute time</th>
<th>2-vertex-reconstruction compute time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>2</td>
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<tr>
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<tr>
<td>6</td>
<td>156</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>1,044</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>12,346</td>
<td>2 min</td>
<td>11 min</td>
</tr>
<tr>
<td>9</td>
<td>274,668</td>
<td>24 min</td>
<td>16.3 hrs</td>
</tr>
<tr>
<td>10</td>
<td>12,005,168</td>
<td>32.8 hrs</td>
<td>9.5 mo ‡</td>
</tr>
<tr>
<td>11</td>
<td>1,018,997,864</td>
<td>1.5 – 2.75 yrs †</td>
<td>1000 yrs ‡</td>
</tr>
<tr>
<td>12</td>
<td>165,091,172,592</td>
<td>300 – 400 yrs ‡</td>
<td>3-5 million yrs ‡</td>
</tr>
</tbody>
</table>

† Brian McMullen’s estimate based on partial computation
‡ Estimate based on non-linear extrapolation
Class Reconstruction Numbers

- Can we reconstruct with fewer cards if we know the graph’s class?
  - $\exists C_{\text{vrn}}$, etc

- Maximal planar graphs have $\exists C_{\text{vrn}}_1 \leq 2$

- Most trees $\exists C_{\text{cern}}_1 = 2$,
  - but there are 6 known to have $\exists C_{\text{cern}}_1 = 3$
Open Questions

• The original Graph Reconstruction Conjecture itself
  – Plus Edge Reconstruction Conjecture
• Do almost all graphs have $\forall vrn_1 = 3$, also?
• What types of graphs always have high $\exists vrn_1$ or $\forall vrn_1$?
• What is the relationship between $\exists vrn_1$ and $\exists ern_1$ for different classes of graphs?
• Are there other trees with $\exists Cern_1 > 2$?
• What is the behavior of $k(>1)$-reconstruction numbers?